

z-Transform

Recall: CT complex exponentials e^{st} w/ $s = \sigma + j\omega$ are eigenfunctions of LTI systems

$$x(t) = e^{s_0 t} \rightarrow \boxed{\text{LTI}} \rightarrow y(t) = H(s_0) e^{s_0 t}$$

$h(t), H(s)$

We can sample the CT sinusoid to determine a similar eigenfunction for DT systems

DT sampling: $X[n] = X(nT_s)$

$$\rightarrow X[n] = e^{snT_s} = (e^{sT_s})^n = (e^{(\sigma + j\omega)T_s})^n = \underline{(e^{\sigma T_s} e^{j\omega T_s})^n}$$

$\rightarrow e^{\sigma T_s} e^{j\omega T_s}$ is a scalar constant that we will call z

$$\cdot |z| = e^{\sigma T_s}, \quad \angle z = \omega T_s$$

$$\rightarrow z = e^{\sigma T_s} e^{j\omega T_s} \rightarrow X[n] = (e^{\sigma T_s} e^{j\omega T_s})^n = \boxed{z^n}$$

A DT signal of the form $X[n] = z^n$ is an eigenfunction of DT LTI systems

$$X[n] = z_0^n \rightarrow \boxed{\text{LTI}} \rightarrow y[n] = \underbrace{H(z_0)}_{\rightarrow H(z)?} z_0^n$$

$h[n], H(z)$

Transfer Function of
DT LTI system
The z-Transform of $h[n]$

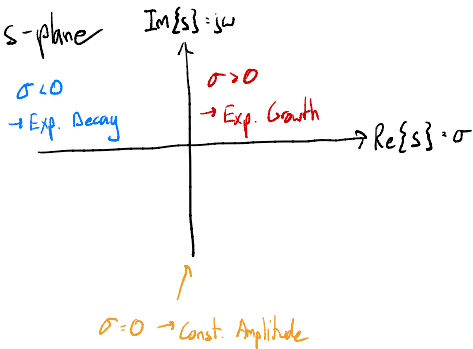
So, if $|z| = e^{\sigma T_s}$, then:

$$\cdot \sigma < 0 \rightarrow |z| < 1$$

$$\cdot \sigma = 0 \rightarrow |z| = 1$$

$$\cdot \sigma > 0 \rightarrow |z| > 1$$

Recall: s-plane



There is an analogous plane in z also $\rightarrow$ **Complex z -plane**

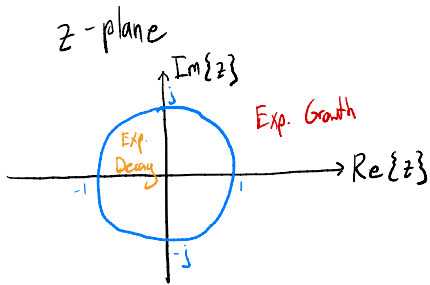
$\rightarrow$ Complex s in e^{st} , real part σ controls growth/decay, imag. part $j\omega$ controls phase rotation

$\rightarrow$ Complex z in z^n , magnitude $|z|$ controls growth/decay, phase $\angle z$ controls phase rotation

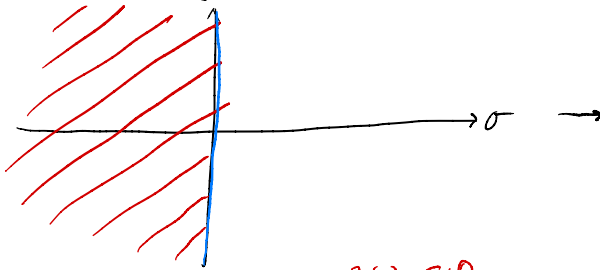
$|z| < 1 \rightarrow$ inside unit circle $\rightarrow$ exp. decay

$|z| = 1 \rightarrow$ on unit circle $\rightarrow$ const. amplitude DT sinusoid

$|z| > 1 \rightarrow$ outside unit circle $\rightarrow$ exp. growth

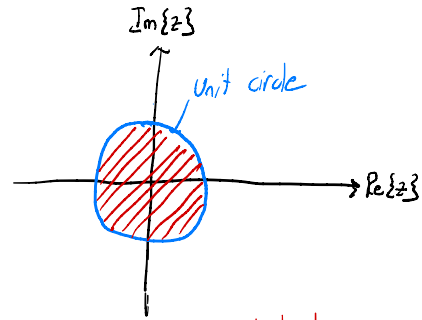
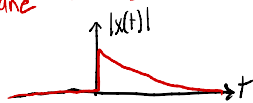


s -plane maps to z -plane:

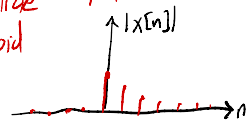


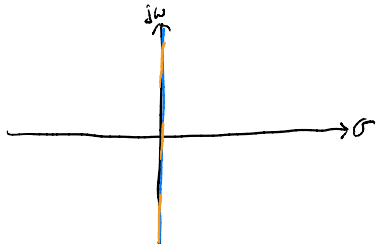
Left side of s -plane $\rightarrow \text{Re}\{s\} = \sigma < 0$

Decaying sinusoid

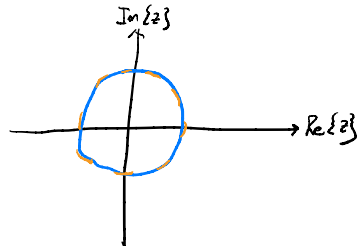
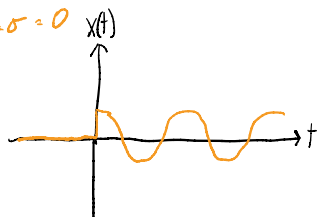


Inside unit circle $\rightarrow |z| < 1$
Decaying sinusoid

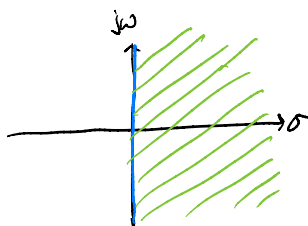
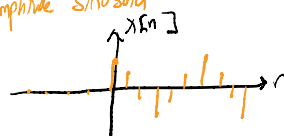




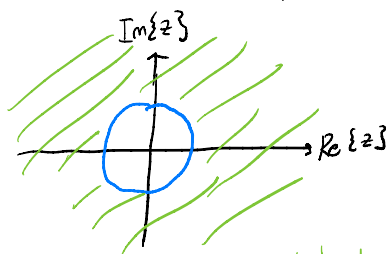
On $j\omega$ -axis $\rightarrow \text{Re}\{s\} = \sigma = 0$
Constant amplitude sinusoid



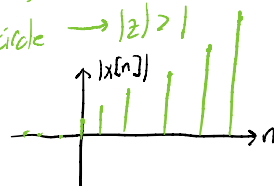
On unit circle $\rightarrow |z| = 1$
Constant amplitude sinusoid



Right side of s -plane $\rightarrow \text{Re}\{s\} = \sigma > 0$
Growing sinusoid



Outside unit circle $\rightarrow |z| > 1$
Growing sinusoid



Recall Laplace transform: $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

• Like a "generalized" Fourier transform — May exist when FT doesn't

• CT $\rightarrow$ won't work w/ DT signals

• We may run into similar situations in DT $\rightarrow$ want to "transform" signals for which the DTFT may not exist

• Try approximating Laplace w/ DT samples w/ sampling period T_s

$$\rightarrow X(s) \approx T_s \sum_{n=-\infty}^{\infty} \underbrace{x(nT_s)}_{= x[n]} e^{-snT_s} = T_s \sum_{n=-\infty}^{\infty} x[n] \underbrace{(e^{sT_s})^{-n}}_{= z^{-n}} = \boxed{\sum_{n=-\infty}^{\infty} x[n] z^{-n} = X(z)}$$

DT "differentiation"
replacing dt from integral

Suppose $T_s = 1$

z -Transform of $x[n]$

Notation: $X(z) = Z\{x[n]\}$

• From above: $X(s) = \lim_{T_s \rightarrow 0} T_s X(z)$

• Laplace transform: Transforms CT signals to complex s-plane

• z-transform: Transforms DT signals to complex z-plane

Recall DTFT: $X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n}$ $\rightarrow$ Notice similarity! $X(f) = X(z) \Big|_{z=e^{j2\pi f}}$
 $\rightarrow$ DTFT is just the z-transform evaluated on the unit circle

- z-transform plays same role as Laplace for DT signals & systems
- May exist when DTFT does not $\rightarrow$ helpful for unstable systems analysis

Relationship to Fourier Analysis

$$\text{Laplace: } X(s) \Big|_{s=j\omega} = \int_{-\infty}^{\infty} x(t) e^{-st} dt \Big|_{s=j\omega} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = X(\omega)$$

$\rightarrow$ CT FT is Laplace evaluated along jw-axis (assumes jw-axis in ROC)

$$\text{z-transform: } X(z) \Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = X(\omega)$$

$\rightarrow$ DTFT is z-transform evaluated along unit circle (assumes unit circle in ROC - more later)

Other interpretation of z-transform:

$$z = r e^{j\omega} \rightarrow |z| = r$$

$$\rightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n] (r e^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \{x[n] r^{-n}\} e^{-j\omega n}$$

$\rightarrow$ z-transform is the DTFT of $x[n] r^{-n}$

$\rightarrow r^{-n}$ is like a "convergence factor" that helps the DTFT exist

Useful Transforms:

	Complex Frequency	"Real" Frequency
CT	Laplace Transform	Fourier Transform ($s = j\omega$) $\leftarrow \text{Re}\{s\} = 0$ axis
DT	z -Transform	DTFT ($z = e^{j\omega}$) unit circle

Note: CT and DT Fourier series are like special cases of the CT and DT Fourier transforms for periodic signals

z -Transform Example: Finite-Duration Signals

$$x_1[n] = \delta[n] + 4\delta[n-1] + 6\delta[n-3] = \{1, 4, 0, 6\}$$

Like DTFT, can easily evaluate w/ direct summation:

$$\rightarrow X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} = x_1[0] z^0 + x_1[1] z^{-1} + x_1[2] z^{-2} + x_1[3] z^{-3} = 1 + 4z^{-1} + 6z^{-3}$$

But at $z=0$, $X_1(z) \rightarrow \infty$

$\rightarrow z=0$ not in region of convergence (ROC) of $X_1(z)$

$\rightarrow$ Like Laplace, z -transform is not complete w/o a ROC!

ROC: all z except $z=0$

Example: $x_2[n] = \{2, 4, 5, 7, 0, 1\}$

$$\rightarrow X_2(z) = \sum_{n=-\infty}^{\infty} x_2[n] z^{-n} = 2z^2 + 4z + 5 + 7z^{-1} + z^{-3}$$

ROC: all z except $z=0$ AND $z=\infty$

Generally: $z\{\delta[n-n_0]\} = z^{-n_0}$ for $n_0 > 0$ and ROC: all $z \neq 0$

$z\{\delta[n+n_0]\} = z^{n_0}$ for $n_0 > 0$ and ROC: all $z \neq \infty$

Example: Right-sided sequence

$$X[n] = a^n u[n] = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

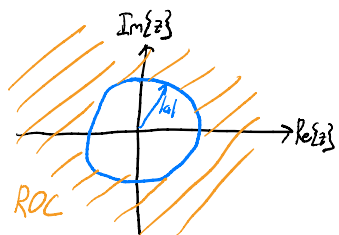
$$\rightarrow X(z) = \sum_{n=-\infty}^{\infty} X[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n$$

→ Power series: only converges if $|a z^{-1}| < 1 \rightarrow |z| > |a|$

→ If $|z| > |a|$, $X(z)$ converges to $X(z) = \frac{1}{1 - a z^{-1}}$

If $|z| < |a|$, $X(z) \rightarrow \infty$; $X(z)$ DNE

$$\rightarrow X[n] = a^n u[n] \xleftrightarrow{z} X(z) = \frac{1}{1 - a z^{-1}}; \text{ROC: } |z| > |a|$$



Left-sided sequence? $X[n] = -a^n u[-n-1] = \begin{cases} 0 & n \geq 0 \\ -a^n & n < 0 \end{cases}$

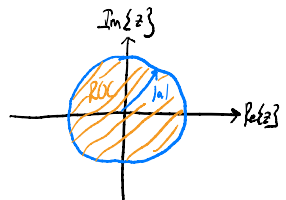
$$\rightarrow X(z) = \sum_{n=-\infty}^{\infty} X[n] z^{-n} = - \sum_{n=-\infty}^{-1} a^n z^{-n} = - \sum_{n=-\infty}^{-1} (a^{-1} z)^{-n} = - \sum_{n=1}^{\infty} (a^{-1} z)^n$$

→ Power series: converges if $|a^{-1} z| < 1 \rightarrow |z| < |a|$

→ If $|z| < |a|$, $X(z)$ converges to $X(z) = \frac{1}{1 - a z^{-1}}$

If $|z| > |a|$, $X(z) \rightarrow \infty$; $X(z)$ DNE

$$\rightarrow X[n] = -a^n u[-n-1] \xleftrightarrow{z} X(z) = \frac{1}{1 - a z^{-1}}; \text{ROC: } |z| < |a|$$



Like Laplace, ROC is required to uniquely specify $X[n]$ from $X(z)$

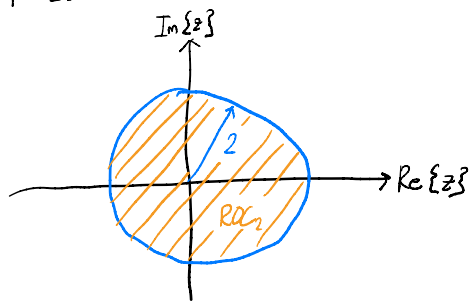
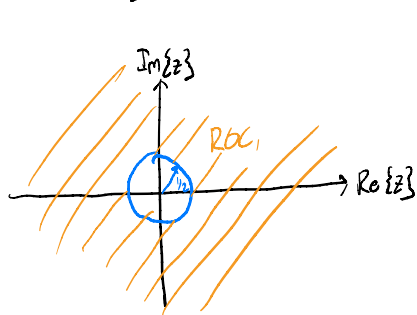
→ ROC: $|z| > |a| \rightarrow X[n] = a^n u[n]$ (right-sided sequence)

$$X(z) = \frac{1}{1 - a z^{-1}}$$

→ ROC: $|z| < |a| \rightarrow X[n] = -a^n u[-n-1]$ (left-sided sequence)

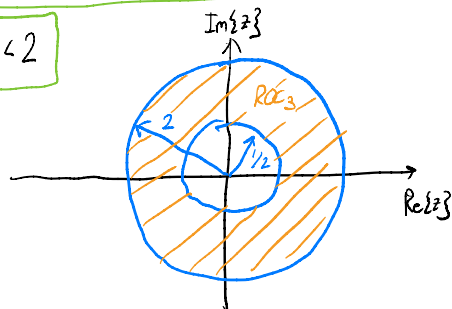
Example: $X_1[n] = \left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{z} X_1(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}; \text{ROC}_1: |z| > \frac{1}{2}$

$X_2[n] = -(2)^n u[-n-1] \xleftrightarrow{z} X_2(z) = \frac{1}{1 - 2z^{-1}}; \text{ROC}_2: |z| < 2$



Let $X_3[n] = \underbrace{X_1[n] + X_2[n]}_{\text{Two-sided!}} \xleftrightarrow{z} X_3(z) = X_1(z) + X_2(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - 2z^{-1}}$

ROC? Overlap of ROC_1 & ROC_2 : $\text{ROC}_3: \frac{1}{2} < |z| < 2$



ROC Properties of z-Transform

- Similar to Laplace

→ For Laplace, ROCs are vertical strips set by boundaries at constant $\text{Re}\{s\} = -a$

→ In z-transform, these map to circles/rings set by boundaries at constant radius $|z| = |a|$ w/ center of circle at origin

- Right-sided $x[n]$: ROC will contain all z where $|z| > |z_0|$ for some z_0

→ Outside the circle of radius $|z_0|$

- Left-sided $x[n]$: ROC will contain all z where $|z| < |z_0|$ for some z_0

→ Inside the circle of radius $|z_0|$

- Two-sided $x[n]$: Divide into right-sided $x_2[n]$ and left-sided $x_1[n]$
ROC will be overlap of the two ROCs (will be an annulus/ring)
No overlap? → ROC is empty & z-transform DNE

- Finite-duration $x[n]$: ROC is entire z-plane except possibly $z=0$ and $z=\infty$

z-Transform Poles and Zeros

Like Laplace $X(s)$, a z-transform $X(z)$ frequently takes a rational form:

$$X(z) = \frac{N(z)}{D(z)} \quad \begin{matrix} \swarrow \text{Polynomials in } z \\ \searrow \end{matrix}$$

DT systems often constructed using linear constant-coefficient difference equations (LCCDEs)
→ give a transfer function of this form

If $N(z_0) = 0 \rightarrow X(z_0) = 0 \rightarrow z_0$ is called a zero of $X(z)$

If $D(z_0) = 0 \rightarrow X(z) \rightarrow \pm\infty \rightarrow z_0$ is called a pole of $X(z)$

Important: ROC of $X(z)$ can contain zeros but cannot contain poles

$N(z)$ and $D(z)$ can be factored into degree -1 polynomials (i.e., $N(z^{-1})$, $D(z^{-1})$)

$$X(z) = \frac{N(z)}{D(z)} = \frac{(1-b_1 z^{-1})(1-b_2 z^{-1}) \cdots (1-b_m z^{-1})}{(1-a_1 z^{-1})(1-a_2 z^{-1}) \cdots (1-a_N z^{-1})} = \frac{\prod_{i=1}^m (1-b_i z^{-1})}{\prod_{j=1}^N (1-a_j z^{-1})}$$

* Note: $az^{-1}|_{z=a} = 1 \rightarrow 1-az^{-1}|_{z=a} = 0$

M zeros & N poles

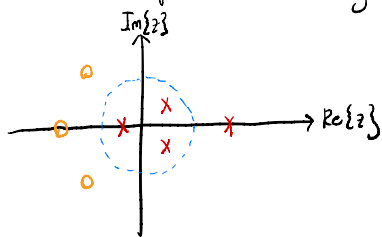
Inverse z-transform of rational $X(z)$? Do PFE to get to the form:

$$X(z) = \frac{A}{1-a_1 z^{-1}} + \frac{B}{1-a_2 z^{-1}} + \dots \quad \xleftrightarrow{z^{-1}} X[n] = A(a_1)^n u[n] + B(a_2)^n u[n] + \dots$$

(or $-A(a_1)^n u[-n-1]$ depending on ROC)

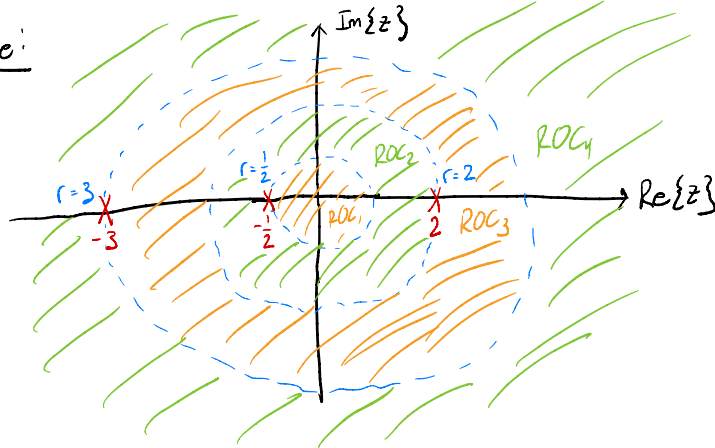
Pole-Zero Plot

Plot of all poles (marked by x's) and zeros (marked by o's) in z-plane



- ROC can contain zeros but cannot contain poles
- Like Laplace, ROC must be outside or inside of all poles, or in between poles
- ROC is a ring, not a vertical strip
- If ROC has $|z| > |a_1|$, the term in $X[n]$ associated with a_1 is right-sided
- If ROC has $|z| < |a_2|$, the term in $X[n]$ associated with a_2 is left-sided

Example:



ROCs:

$$ROC_1: |z| < \frac{1}{2} \rightarrow X[n] = -A\left(-\frac{1}{2}\right)^n u[-n-1] - B(2)^n u[-n-1] - C(-3)^n u[-n-1] \rightarrow \text{Left-sided}$$

$$ROC_2: \frac{1}{2} < |z| < 2 \rightarrow X[n] = A\left(-\frac{1}{2}\right)^n u[n] - B(2)^n u[-n-1] - C(-3)^n u[-n-1] \rightarrow \text{Two-sided}$$

$$ROC_3: 2 < |z| < 3 \rightarrow X[n] = A\left(-\frac{1}{2}\right)^n u[n] + B(2)^n u[n] - C(-3)^n u[-n-1] \rightarrow \text{Two-sided}$$

$$ROC_4: 3 < |z| \rightarrow X[n] = A\left(-\frac{1}{2}\right)^n u[n] + B(2)^n u[n] + C(-3)^n u[n] \rightarrow \text{Right-sided}$$

Properties of the z-Transform:

1. Linearity: $x[n] = a_1 x_1[n] + a_2 x_2[n] \xrightarrow{z} X(z) = a_1 X_1(z) + a_2 X_2(z)$ ROC: Intersection of ROC_1 & ROC_2

2. Time-Shifting: $x[n-n_0] \xrightarrow{z} z^{-n_0} X(z)$ ROC: New ROC may add or delete $z=0, \infty$

3. Convolution: $y[n] = x[n] * h[n] \xrightarrow{z} Y(z) = \underbrace{X(z)}_{\text{Input}} \underbrace{H(z)}_{\text{System}}$ ROC: At least intersection of ROC_1 & ROC_2
May be larger if some poles canceled by zeros
"Transfer Function" $\rightarrow H(z) = Z\{h[n]\}$

For unilateral z-transform:
$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

Initial value theorem: $x[0] = \lim_{z \rightarrow \infty} X(z)$

Time-Delay: $x[n-1] \xrightarrow{z} z^{-1} X(z) + \underbrace{x[-1]}_{\text{Initial condition (IC)}}$

Double Time-Delay: $x[n-2] \xrightarrow{z} z^{-2} X(z) + z^{-1} x[-1] + x[-2]$

$\rightarrow$ This property is analogous to Laplace time-differentiation property

Inverse z-Transform

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \rightarrow \text{Contour integration around a circle } C \text{ within ROC}$$

$\rightarrow$ Req's complex analysis $\rightarrow$ We just use table

Solving LCCDEs

Linear Constant Coefficient Difference Equations (LCCDEs) (similar to CT LCCDEs)

$$a_0 y[n] + a_1 y[n-1] + a_2 y[n-2] + \dots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \dots + b_M x[n-M]$$

$$\rightarrow \sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k] \rightarrow y[n] \text{ output, } x[n] \text{ input} \rightarrow \text{LCCDEs describe systems!}$$

$\rightarrow$ Any DT system described by an LCCDE is LTI
e.g., digital filters

Example:

$$y[n] + 4.5y[n-1] + 2y[n-2] = 3.25x[n] + 4.25x[n-1]$$

Initial Conditions: $y[-1] = 1$, $y[-2] = -3.75$ (Note: we need N IC's)

Input: $x[n] = u[n]$ (Note: we will assume $x[n]$ IC's = 0)

Recall: $y[n] \xleftrightarrow{z} Y(z)$, $y[n-1] \xleftrightarrow{z} z^{-1}Y(z) + y[-1]$, $y[n-2] \xleftrightarrow{z} z^{-2}Y(z) + z^{-1}y[-1] + y[-2]$

$\rightarrow$ Apply z-transform to both sides of equation to get expression in z^{-1} , $Y(z)$, and $X(z)$

$$\rightarrow y[n] \xleftrightarrow{z} Y(z)$$

$$4.5y[n-1] \xleftrightarrow{z} 4.5z^{-1}Y(z) + 4.5y[-1]$$

$$2y[n-2] \xleftrightarrow{z} 2z^{-2}Y(z) + 2z^{-1}y[-1] + 2y[-2]$$

$$3.25x[n] \xleftrightarrow{z} 3.25X(z)$$

$$4.25x[n-1] \xleftrightarrow{z} 4.25z^{-1}X(z) + 4.25\underbrace{x[-1]}_{=0} = 4.25z^{-1}X(z)$$

$$\rightarrow Y(z) + 4.5z^{-1}Y(z) + 4.5y[-1] + 2z^{-2}Y(z) + 2z^{-1}y[-1] + 2y[-2] = 3.25X(z) + 4.25z^{-1}X(z)$$

$$\rightarrow (1 + 4.5z^{-1} + 2z^{-2})Y(z) + (\underbrace{4.5y[-1]}_{=1} + \underbrace{2y[-2]}_{=-3.75}) + \underbrace{2z^{-1}y[-1]}_{=1} = (3.25 + 4.25z^{-1})X(z)$$

$$\rightarrow \boxed{(1 + 4.5z^{-1} + 2z^{-2})Y(z) - \underbrace{3 + 2z^{-1}}_{\text{IC's}} = (3.25 + 4.25z^{-1})X(z)}$$

Solve DT LCCDEs same as CT w/ Laplace:

1.) Use time-delay property of z -transform to get:

$$\underbrace{Q(z)Y(z)}_{\text{From } y[n] \text{ coefficients}} + \underbrace{f(z)}_{\text{From IC's}} = \underbrace{P(z)X(z)}_{\text{From } x[n] \text{ coefficients}}$$

2.) For total solution $y[n]$: Rearrange to $Y(z) = \frac{P(z)X(z) - f(z)}{Q(z)}$

• Do PFE, then inverse z -transform to get $y[n]$

• Can also do $y[n] = y_{zs}[n] + y_{zi}[n]$

3.) For "zero-state" solution $y_{zs}[n]$

• Same as #2 but set IC's = 0 (i.e., $f(s) = 0$) $\rightarrow$ solve using given $x[n]$

4.) For "zero-input" solution $y_{zi}[n]$

• Same as #2 but set $X(z) = 0$, only use IC's

5.) For transfer function $H(z)$ or impulse response $h[n]$

• Same as #2 but set IC's = 0 and $X(z) = 1$

$$H(z) = \frac{Y(z)}{X(z)}$$

Zero-State Solution $y_{zs}[n]$: IC's = 0 and use input $x[n]$

$$x[n] = u[n] \xrightarrow{z} X(z) = \frac{1}{1-z^{-1}}; \text{ROC: } |z| > 1$$

$$\begin{aligned} \rightarrow (1 + 4.5z^{-1} + 2z^{-2})Y(z) &= \frac{3.25 + 4.25z^{-1}}{1-z^{-1}} \rightarrow Y(z) = \frac{3.25 + 4.25z^{-1}}{(1-z^{-1})(1+\frac{1}{2}z^{-1})(1+4z^{-1})} \\ &= z^{-2}(z^2 + 4.5z + 2) \end{aligned}$$

Can use quadratic equation from here if necessary

$$A = (1-z^{-1})Y(z) \Big|_{z=1} = \frac{3.25 + 4.25}{(1+\frac{1}{2})(1+4)} = \frac{7.5}{(1.5)(5)} = 1$$

$$B = (1+\frac{1}{2}z^{-1})Y(z) \Big|_{z=-\frac{1}{2}} = \frac{3.25 + 4.25(-2)}{(1+2)(1+4(-2))} = \frac{3.25 - 8.5}{(3)(-7)} = \frac{-5.25}{-21} = \frac{-21/4}{-21} = \underline{\underline{\frac{1}{4}}}$$

$$C = (1 + 4z^{-1})Y(z) \Big|_{z=-4} = \frac{3 \cdot 25 + 4 \cdot 25(-\frac{1}{4})}{(1 + \frac{1}{4})(1 + \frac{1}{2}(-\frac{1}{4}))} = \frac{\frac{13}{4} - \frac{17}{4} \cdot \frac{1}{4}}{\frac{5}{4} \cdot \frac{7}{8}} = \frac{\frac{52}{16} - \frac{17}{16}}{\frac{35}{32}} = \frac{\frac{35}{16}}{\frac{35}{32}} = 2$$

$$\rightarrow Y(z) = \frac{1}{1-z^{-1}} + \frac{\frac{1}{4}}{1+\frac{1}{2}z^{-1}} + \frac{2}{1+4z^{-1}} \xrightarrow{z} \boxed{y_{zs}[n] = u[n] + \frac{1}{4}(-\frac{1}{2})^n u[n] + 2(-4)^n u[n]}$$

* Like Laplace, generally assume output signals are causal due to unilateral z-transform

Zero-Input Solution $y_{zi}[n]$: $X(z) = 0$

$$(1 + 4.5z^{-1} + 2z^{-2})Y(z) \stackrel{\text{IC's}}{=} -3 + 2z^{-1} = 0 \rightarrow (1 + 4.5z^{-1} + 2z^{-2})Y(z) = 3 - 2z^{-2}$$

$$\rightarrow Y(z) = \frac{3 - 2z^{-1}}{1 + 4.5z^{-1} + 2z^{-2}} = \frac{3 - 2z^{-1}}{(1 + \frac{1}{2}z^{-1})(1 + 4z^{-1})} = \frac{A}{1 + \frac{1}{2}z^{-1}} + \frac{B}{1 + 4z^{-1}}$$

$$A = (1 + \frac{1}{2}z^{-1})Y(z) \Big|_{z=-\frac{1}{2}} = \frac{3 - 2(-2)}{1 + 4(-2)} = \frac{3 + 4}{1 - 8} = \frac{7}{-7} = -1$$

$$B = (1 + 4z^{-1})Y(z) \Big|_{z=-4} = \frac{3 - 2(-\frac{1}{4})}{1 + \frac{1}{2}(-\frac{1}{4})} = \frac{3 + \frac{1}{2}}{1 - \frac{1}{8}} = \frac{\frac{7}{2}}{\frac{7}{8}} = 4$$

$$\rightarrow Y(z) = -\frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{4}{1 + 4z^{-1}} \rightarrow \boxed{y_{zi}[n] = -(-\frac{1}{2})^n u[n] + 4(-4)^n u[n]}$$

Total Solution $y[n]$: $\boxed{y[n] = y_{zs}[n] + y_{zi}[n] = u[n] - \frac{3}{4}(-\frac{1}{2})^n u[n] + 6(-4)^n u[n]}$

Impulse Response $h[n]$: IC's = 0, $H(z) = \frac{Y(z)}{X(z)} \leftarrow \text{Transfer Function}$

$$(1 + 4.5z^{-1} + 2z^{-2})Y(z) = (3.25 + 4.25z^{-1})X(z) \rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{3.25 + 4.25z^{-1}}{(1 + \frac{1}{2}z^{-1})(1 + 4z^{-1})}$$

$$= \frac{A}{1 + \frac{1}{2}z^{-1}} + \frac{B}{1 + 4z^{-1}}$$

$$A = (1 + \frac{1}{2}z^{-1})H(z) \Big|_{z=-\frac{1}{2}} = \frac{3.25 + 4.25(-2)}{1 + 4(-2)} = \frac{3.25 - 8.5}{1 - 8} = \frac{-5.25}{-7} = \frac{3}{4}$$

$$H(z) = \frac{-3/4}{1 + \frac{1}{2}z^{-1}} + \frac{5/2}{1 + 4z^{-1}}$$

$$B = (1 + 4z^{-1})H(z) \Big|_{z=-4} = \frac{3.25 + 4.25(-\frac{1}{4})}{1 + \frac{1}{2}(-\frac{1}{4})} = \frac{\frac{13}{4} - \frac{17}{4} \cdot \frac{1}{4}}{1 - \frac{1}{8}} = \frac{\frac{52}{16} - \frac{17}{16}}{\frac{7}{8}} = \frac{\frac{35}{16}}{\frac{7}{8}} = \frac{35}{2 \cdot 7} = \frac{5}{2}$$

$h[n]$ depends on ROC!

ROC: $|z| > 4$

For now, assume causal $h[n] \rightarrow \boxed{h[n] = -\frac{3}{4}(-\frac{1}{2})^n u[n] + \frac{5}{2}(-4)^n u[n]}$

DT System Analysis's Using z-Transform

For transfer function: IC's = 0

LCCDE (LTI system): $\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^N b_k x[n-k] \xleftrightarrow{z} Y(z) \sum_{k=0}^N a_k z^{-k} = X(z) \sum_{k=0}^M b_k z^{-k}$

$\rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$

Annotations:

- Polynomial of $z \rightarrow$ gives zeros (points to numerator)
- Rational $H(z)$ (points to the fraction)
- Polynomial of $z \rightarrow$ gives poles (points to denominator)

$\rightarrow$ Convolution: $y[n] = x[n] * h[n] \xleftrightarrow{z} Y(z) = X(z)H(z)$

If $e^{j\omega}$ (unit circle, $|z|=1$) in ROC, then

$y[n] = x[n] * h[n] \xleftrightarrow{F} Y(\omega) = X(\omega)H(\omega) \text{ or } Y(f) = X(f)H(f)$

Eigenfunction property: $x[n] = z_0^n \xrightarrow{H} y[n] = H(z_0) z_0^n$

Problem: We need extra information from system to determine ROC and $h[n]$

$\rightarrow$ causality and stability

Causality: $h[n] = 0$ for $n < 0 \rightarrow h[n]$ is right-sided

$\rightarrow$ ROC of $H(z)$ must be exterior of circle and include $z = \infty$

If $H(z)$ rational, ROC must be outer circle and $N(z)$ lower order than $D(z)$

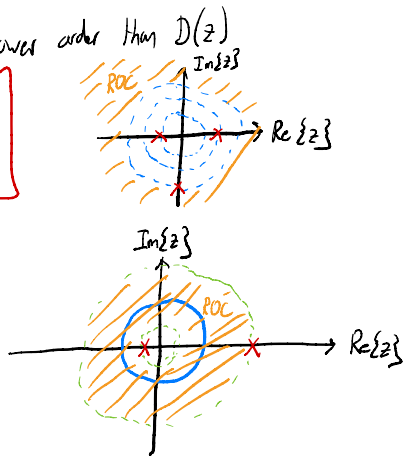
A rational $H(z)$ can only be causal if the ROC is the exterior of a circle and includes $z = \infty$

BIBO Stability: Stability criterion: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

$\rightarrow$ Implies that the DTFT $H(e^{j\omega})$ or $H(\omega)$ exists

$\rightarrow$ The ROC must include unit circle

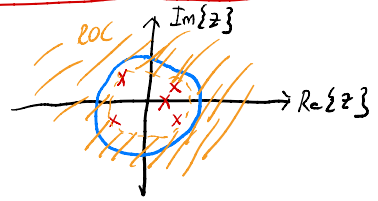
A DT LTI system is stable if and only if the ROC of $H(z)$ includes the unit circle



Causal & Stable Systems

Causal $\rightarrow$ ROC exterior circle
Stable $\rightarrow$ ROC includes unit circle

A causal DT LTI system is stable if and only if all its poles are within unit circle
i.e., magnitude of all poles is less than 1

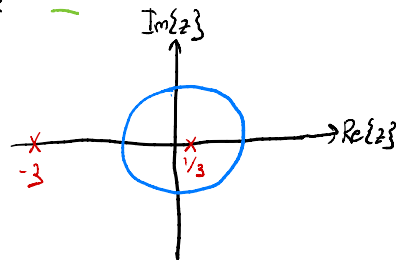


Example: $H(z) = \frac{3 + \frac{17}{3}z^{-1}}{1 + \frac{8}{3}z^{-1} - z^{-2}} = \frac{3 + \frac{17}{3}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 + 3z^{-1})} = \frac{A}{1 - \frac{1}{3}z^{-1}} + \frac{B}{1 + 3z^{-1}}$

$$A = (1 - \frac{1}{3}z^{-1})H(z)|_{z=\frac{1}{3}} = \frac{3 + \frac{17}{3}(3)}{1 + 3(3)} = \frac{3 + 17}{1 + 9} = \frac{20}{10} = 2$$

$$B = (1 + 3z^{-1})H(z)|_{z=-3} = \frac{3 + \frac{17}{3}(-\frac{1}{3})}{1 - \frac{1}{3}(-\frac{1}{3})} = \frac{\frac{27}{9} - \frac{17}{9}}{1 + \frac{1}{9}} = \frac{\frac{10}{9}}{\frac{10}{9}} = 1$$

$\rightarrow H(z) = \frac{2}{1 - \frac{1}{3}z^{-1}} + \frac{1}{1 + 3z^{-1}} \rightarrow$ Poles at $z = \frac{1}{3}, -3$



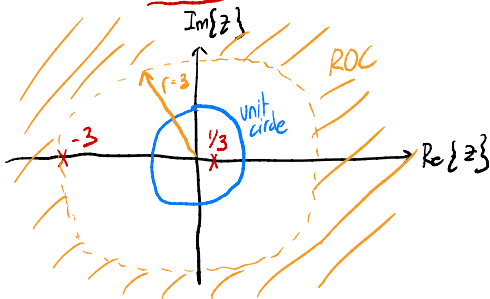
Can system be both causal & stable?

$\rightarrow$ No, because there is a pole outside unit circle ($|1/3| > 1$)

$h[n]$ if system is causal? $\rightarrow$ ROC: $|z| > 3 \rightarrow h[n] = 2(\frac{1}{3})^n u[n] + (-3)^n u[n]$

$h[n]$ if system is stable? $\rightarrow$ ROC: $\frac{1}{3} < |z| < 3 \rightarrow h[n] = 2(\frac{1}{3})^n u[n] - (-3)^n u[-n-1]$

Causal:



Stable:

