Transformations of the Independent Variable (Chapter 1.2)

Often want to operate on signals by transforming independent variable

- e.g., delay/advance, stretch/compress, time-reversal

Suppose we have a signal $x(t)$

t ← independent variable

Define a new signal: $\phi(t) = A\,x(at - b)$

A: scales amplitude. Does not affect independent variable

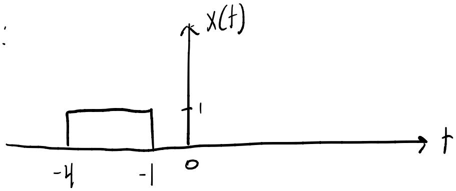
a: stretches or compresses signal – if $a < 0$, also time-reverses signal

b: shifts signal left (advance) or right (delay)

Steps for solving:

1. Factor into $\phi(t) = x(a(t - b/a)) = x(a(t - t_0)) \rightarrow t_0 = b/a$
2. If $a < 0$, time-reverse / flip the signal about $t = 0$
3. If $|a| \neq 1$, then expand or compress
   i. If $|a| > 1 \rightarrow$ Compress
   ii. If $|a| < 1 \rightarrow$ Expand
4. Shift by $t_0 = b/a$
   - If $t_0 > 0$ (e.g., $(t-3)$), then shift right; if $t_0 < 0$ (e.g., $(t+3)$), then shift left
5. Verify result by checking <u>key points</u>

Example:

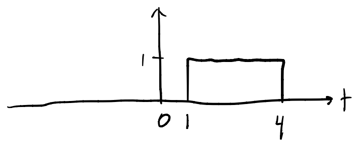


Find and plot $\phi(t) = x(-0.5t - 3)$

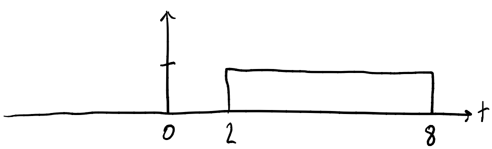
1. Factor: $\phi(t) = x\left(-0.5\left(t - {}^{3}/_{(-0.5)}\right)\right) = x(-0.5(t+6))$

   $t_0 = -6$

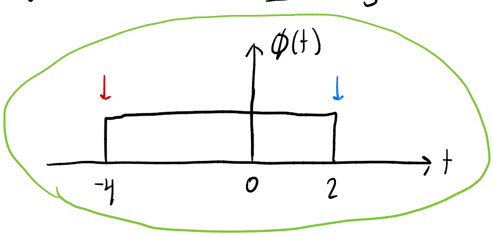
2. $a < 0$, so time reverse:



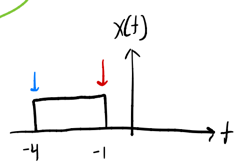
3. $|a| < 0$ so <u>expand</u> or <u>stretch</u> by factor of 2



4. $t_0 < 0$, so shift <u>left</u> by 6 units



5. Verify with key points:



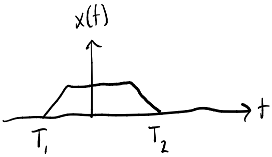
$\phi(-4) = x(-0.5(-4)-3) = x(-1)$ ✓

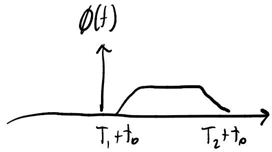
$\phi(2) = x(-0.5(2)-3) = x(-4)$ ✓

Special Case #1: $a = 1$

$$\phi(t) = x(a(t - t_0)) = x(t - t_0)$$

"Time-shifting"





$t_0 > 0 \rightarrow \phi(t)$ is "delayed"

$t_0 < 0 \rightarrow \phi(t)$ is "advanced"

Discrete case: $\phi[n] = x[n - n_0]$

n_0 <u>must</u> be an integer

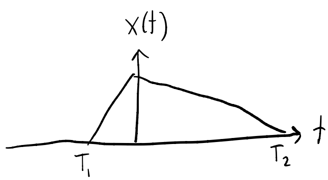
$n_0 > 0 \rightarrow \phi[n]$ is "delayed"

$n_0 < 0 \rightarrow \phi[n]$ is "advanced"

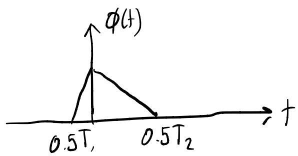
Special Case #2: $t_0 = 0,\ a > 0$

$$\phi(t) = x(a(t - t_0)) = x(at)$$

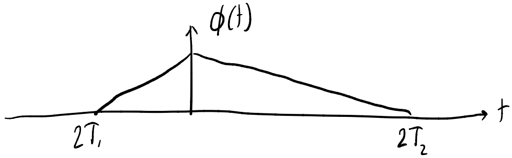
"Time-Scaling"



- If $a > 1$, then $\phi(t)$ is a times "faster" → "Compression"



- If $a<1$, then $\phi(t)$ is a times "slower" → "Expansion"



Discrete case is more complicated

$\phi[n] = x[an]$

- If $a>1$, samples are skipped "Decimation"

  e.g., $a=2 \rightarrow \phi[0] = x[0],\ \phi[1] = x[2],$ ----

- If $a<1$, need samples that are undefined

  e.g., $a = 0.5 \rightarrow \phi[0] = x[0],\ \phi[1] = x[0.5],\ \phi[2] = x[1],$ .... "Interpolation"

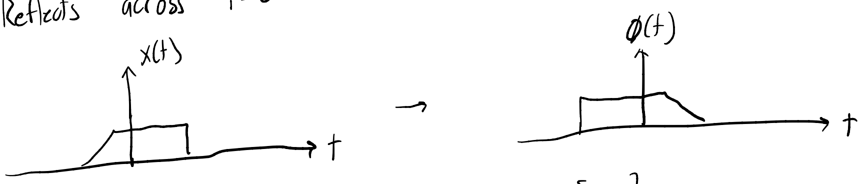
- Considered in DSP

Special Case #3: $t_0 = 0,\ a = -1$

$\phi(t) = x(a(t-t_0)) = x(-t)$

"Time-Reversal"

Reflects across $t=0$



Same result in discrete case: $\phi[n] = x[-n]$

Example: $x(t) = e^{-3t}\cos(\pi t)\big(u(t) - u(t-4)\big)$

Decaying

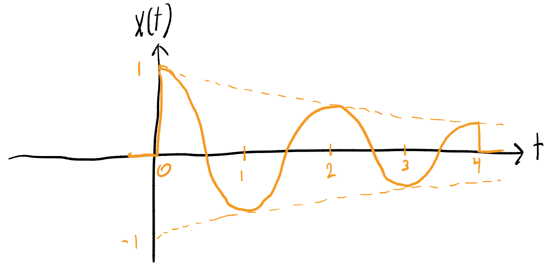
$\cos(2\pi F_0 t)$

$\rightarrow F_0 = \frac{1}{2} \rightarrow T_0 = 2$

On at $t = 0$

Off at $t = 4$

Sketch $x(t)$.



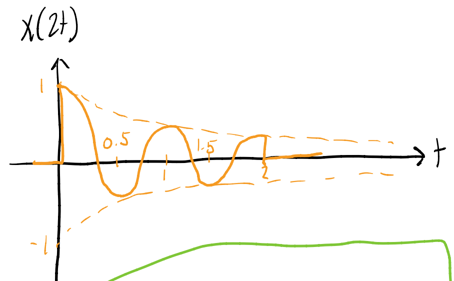
Sketch $x_2(t) = x(2t+2)$.

1.) Factor to $x(a(t - t_0)) \longrightarrow a = 2,\ t_0 = \frac{b}{a} = \frac{-2}{2} = -1$

$\longrightarrow \phi(t) = x(2(t+1))$

2.) Flip $\rightarrow$ not necessary because $a > 0$

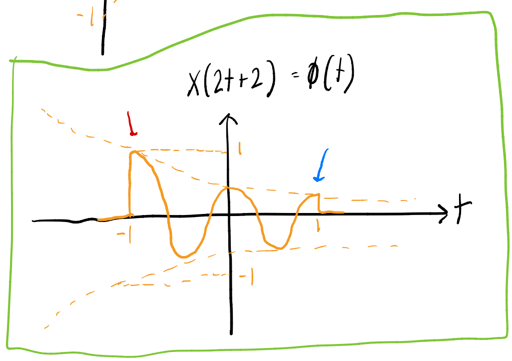
3.) $|a| > 1$ so compress by factor of 2 $\longrightarrow$

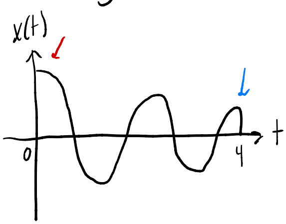


4.) $t_0 < 0$ so shift left by 1

5.) Verify key points:

$\longrightarrow$





$\phi(-1) = x(2(-1)+2) = x(0)$ ✓

$\phi(1) = x(2(1)+2) = x(4)$ ✓

Example: $x[n] = \left(\frac{1}{2}e^{j\frac{\pi}{2}}\right)^n \left(u[n+2] - u[n-2]\right)$ Sketch the real part of $x[n]$.

Rewrite: $x[n] = \left(\frac{1}{2}\right)^n e^{j\frac{\pi}{2}n}\left(u[n+2] - u[n-2]\right)$

Decaying

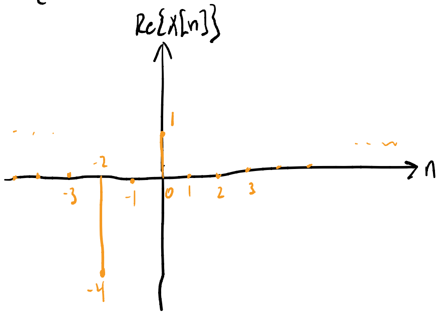
$= \cos\left(\frac{\pi}{2}n\right) + j\sin\left(\frac{\pi}{2}n\right)$

Rotates $\frac{\pi}{2}$ rad per sample

On at $n = -2$

Off at $n = 2$

$\rightarrow \text{Re}\{x[n]\} = \left(\frac{1}{2}\right)^n \cos\left(\frac{\pi}{2}n\right)\left(u[n+2] - u[n-2]\right)$



$x[-3] = 0$

$x[-2] = 4\cos(-\pi) = -4$

$x[-1] = 2\cos\left(-\frac{\pi}{2}\right) = 0$

$x[0] = \cos(0) = 1$

$x[1] = \frac{1}{2}\cos\left(\frac{\pi}{2}\right) = 0$

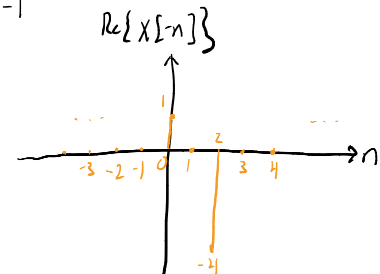
$x[2] = 0$

Sketch the real part of $x[-n-2]$.

1.) Factor to $x[a(n-n_0)] \longrightarrow a = -1,\ t_0 = \frac{b}{a} = \frac{-2}{-1} = 2$

$\longrightarrow \phi[n] = x[-(n+2)]$.

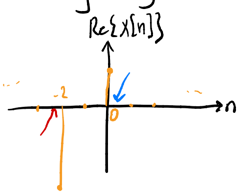
2.) Time-Reverse/Flip, because $a < 0$ $\longrightarrow$



3.) $|a| = 1$, so no compression/expansion

4.) $n_0 = -2$, so shift left by 2

5. Verify key points:



$\phi[0] = x[0-2] = x[-2]$ ✓

$\phi[-2] = x[2-2] = x[0]$ ✓