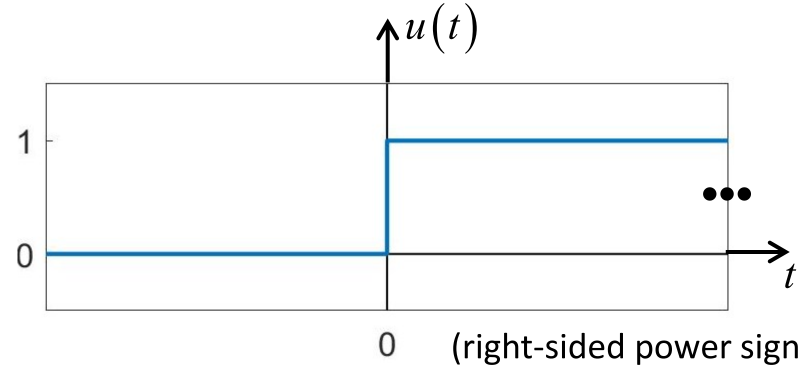
Important Signal Types

ECE 3793

CT Unit Step Function

- Continuous version:

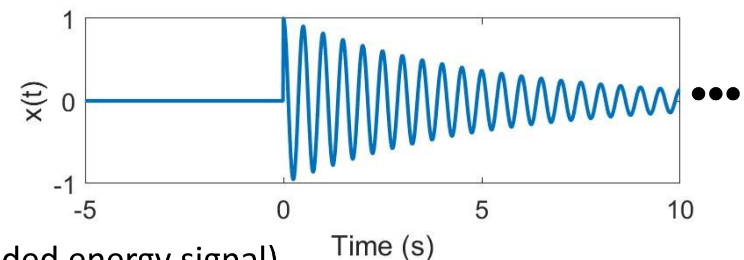
$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



(right-sided power signal)

  - Useful for step-wise and piecewise descriptions
  - For example,

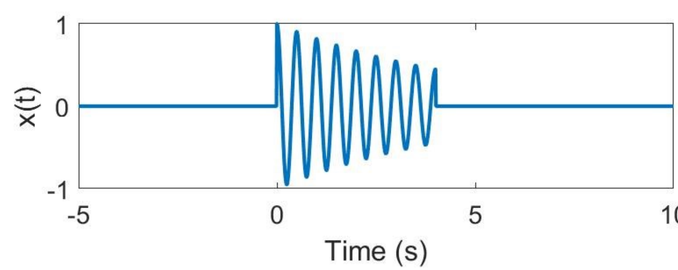
$$x(t) = \cos(\omega_0 t)\exp(-0.2t)u(t)$$



(right-sided energy signal)

$$x(t) = \cos(\omega_0 t)\exp(-0.2t)\big(u(t) - u(t-4)\big)$$

$$= \begin{cases} 0 & t < 0 \\ \cos(\omega_0 t)\exp(-0.2t) & 0 \leq t < 4 \\ 0 & t \geq 4 \end{cases}$$



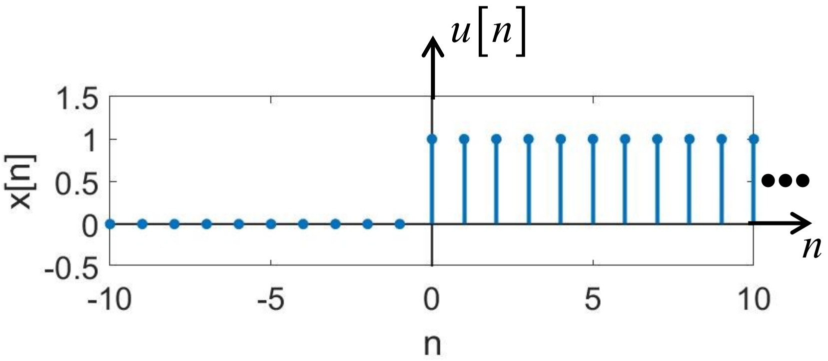
(finite-duration energy signal)

Note: *u*(*t*) is *not* the default Heaviside function in Matlab

DT Unit Step Sequence

- Discrete-Time Version

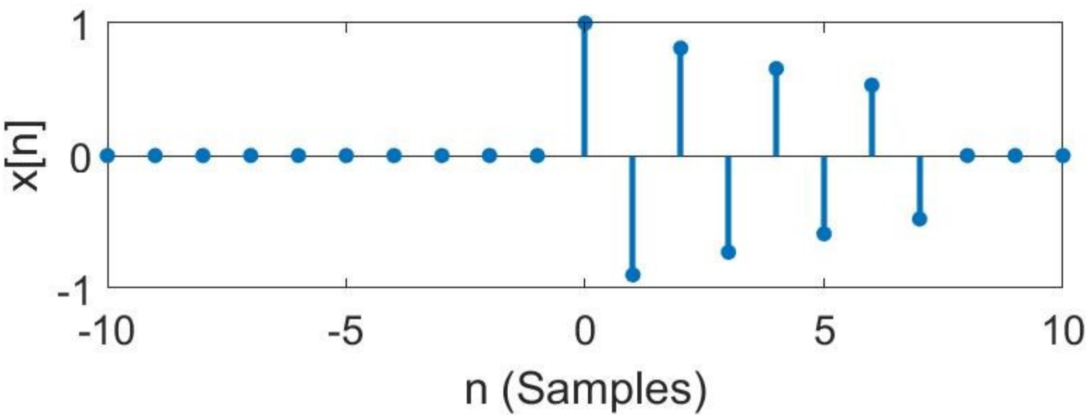
$$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$



  - again, useful for step-wise/piecewise definitions

$$x[n] = (-0.9)^n (u[n] - u[n-8])$$

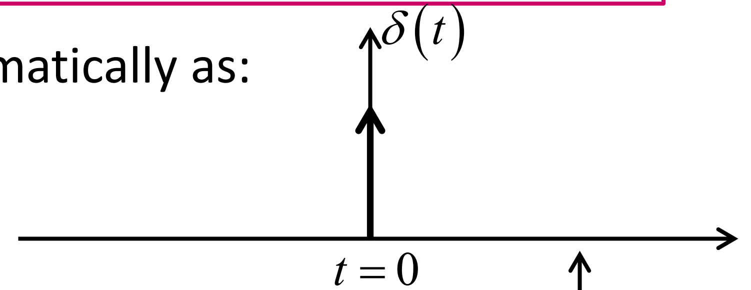
$$= \begin{cases} 0 & n < 0 \\ (-0.9)^n & 0 \leq n \leq 7 \\ 0 & n \geq 8 \end{cases}$$



Important: subtracting u[n-8] ensures that the signal is <u>off</u> at n=8!

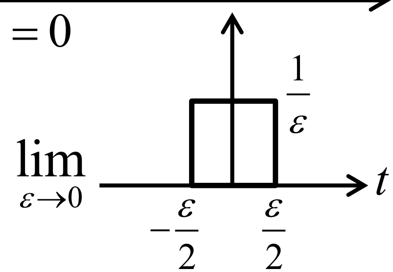
Dirac Delta Function

- Continuous version is defined mathematically as:



$$\delta(t)=0;\ t\neq 0 \quad \text{and} \quad \int_{-\infty}^{\infty}\delta(t)\,dt=1$$

- Infinitely tall, infinitely narrow, with an area = 1
- Can think of as a narrow rectangle, having height inversely proportional to width, and let width go to zero:

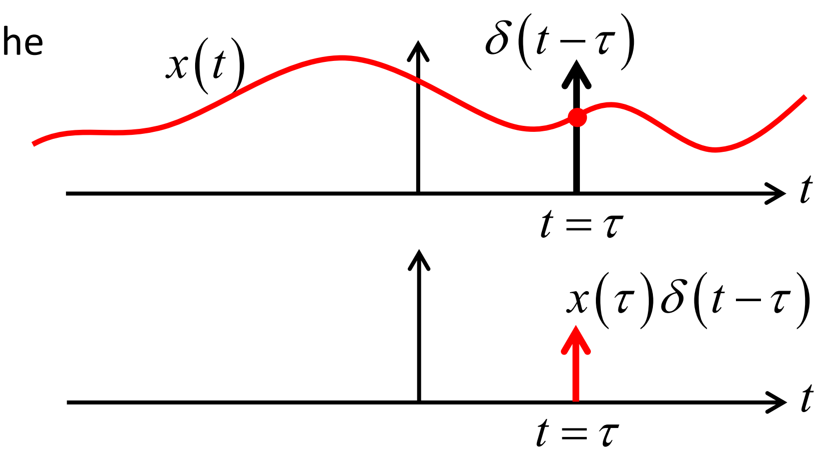


- Critical property: Sifting property or Sampling property

$$x(t)\delta(t)=x(0)\delta(t) \qquad \int_{-\infty}^{\infty}x(t)\delta(t)\,dt=x(0)\int_{-\infty}^{\infty}\delta(t)\,dt=x(0)$$

or… $x(t)\delta(t-T)=x(T)\delta(t-T)$ $\qquad \int_{-\infty}^{\infty}x(t)\delta(t-T)\,dt=x(T)\int_{-\infty}^{\infty}\delta(t-T)\,dt=x(T)$

- Notice how the delta function "samples" the signal at the time instant where the delta function is non-zero
  - (as long as the delta function is located within the limits of integration)
- Example:

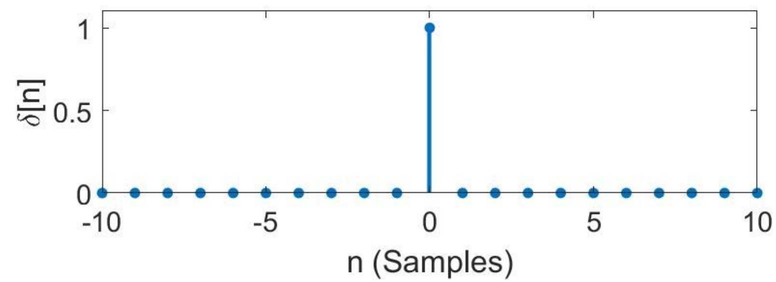


$$\int_{-5}^{5}\cos(2\pi t)\delta(t-1)dt=? \qquad \int_{-5}^{5}\cos(2\pi t)\delta(t-6)dt=?$$

Unit Impulse Function

- Discrete-time version of an impulse

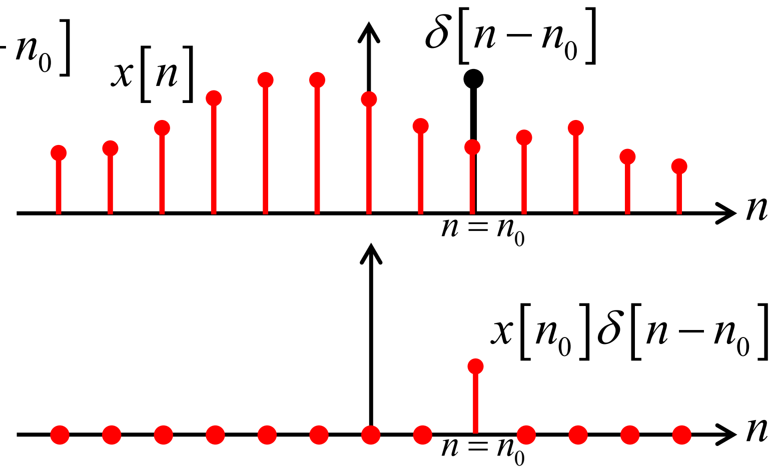
$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$



  - Width or area is meaningless because independent variable (horizontal axis) is not continuous

  - Note similar properties to the sampling property:

$x[n]\delta[n] = x[0]\delta[n]$ and $x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$



$$\sum_{n=-\infty}^{\infty} x[n]\delta[n] = x[0]\sum_{n=-\infty}^{\infty}\delta[n] = x[0]$$ and

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n-n_0] = x[n_0]\sum_{n=-\infty}^{\infty}\delta[n-n_0] = x[n_0]$$

  - Note we can express a sequence as:

$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \ldots$$

$$= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

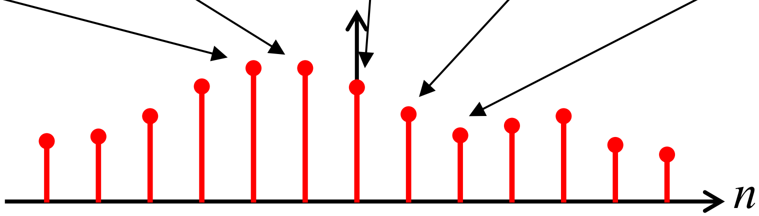
Is there a continuous-time version of this property…?

Functions as Sums of Impulses

- A DT sequence can be expressed as a sum of scaled and shifted unit impulse functions

$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \ldots$$

Each point in the sequence is an individual unit impulse



$$= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- As we switch back and forth between DT and CT, oftentimes similar properties hold in each domain
  - Discrete sums in DT usually just become integrals in CT

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

The summation index k is swapped for the variable of integration τ

Just like in DT, a CT function is an infinite, continuous sum (integral) of scaled, shifted Dirac delta functions!

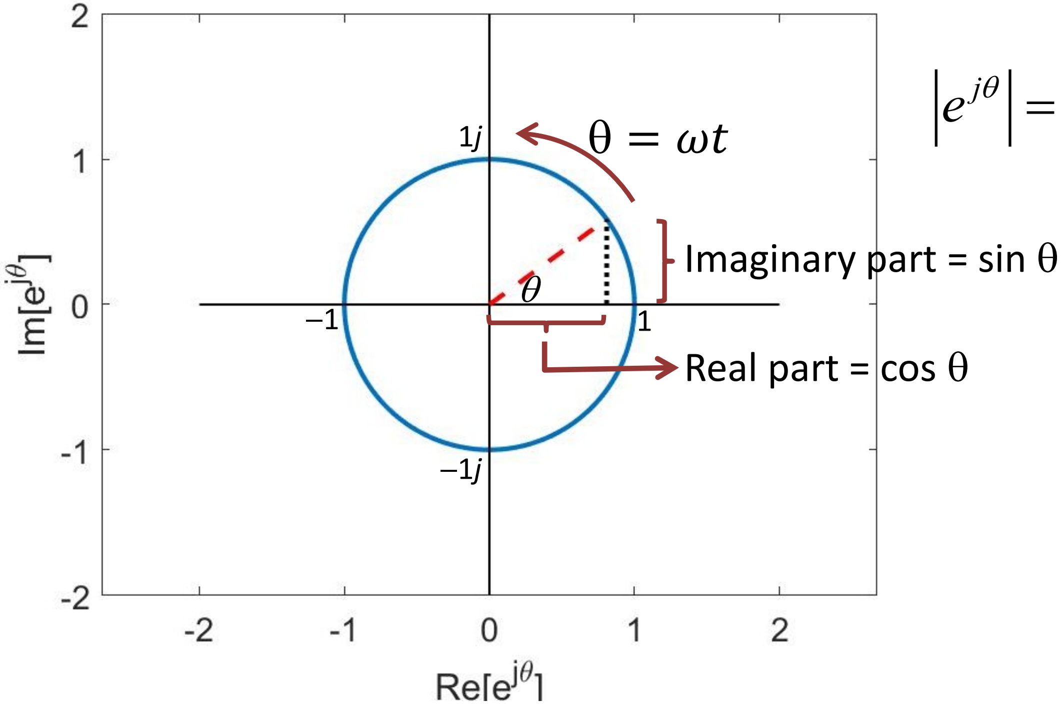
$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau = x(t)\int_{-\infty}^{\infty} \delta(t-\tau)d\tau = \boxed{x(t)}$$

- We could prove this more rigorously…
  - … ~~but I can't be bothered~~ but this is way outside the scope of this class
- This property is very abstract and we won't focus too much on it, but it is very important for the definition of convolution

Complex Sinusoid

$x(t) = \exp(j\omega t)$ t in seconds; ω in radians/second

- Recall that: $e^{j\theta} = \cos\theta + j\sin\theta$ (can be derived via Taylor expansion)



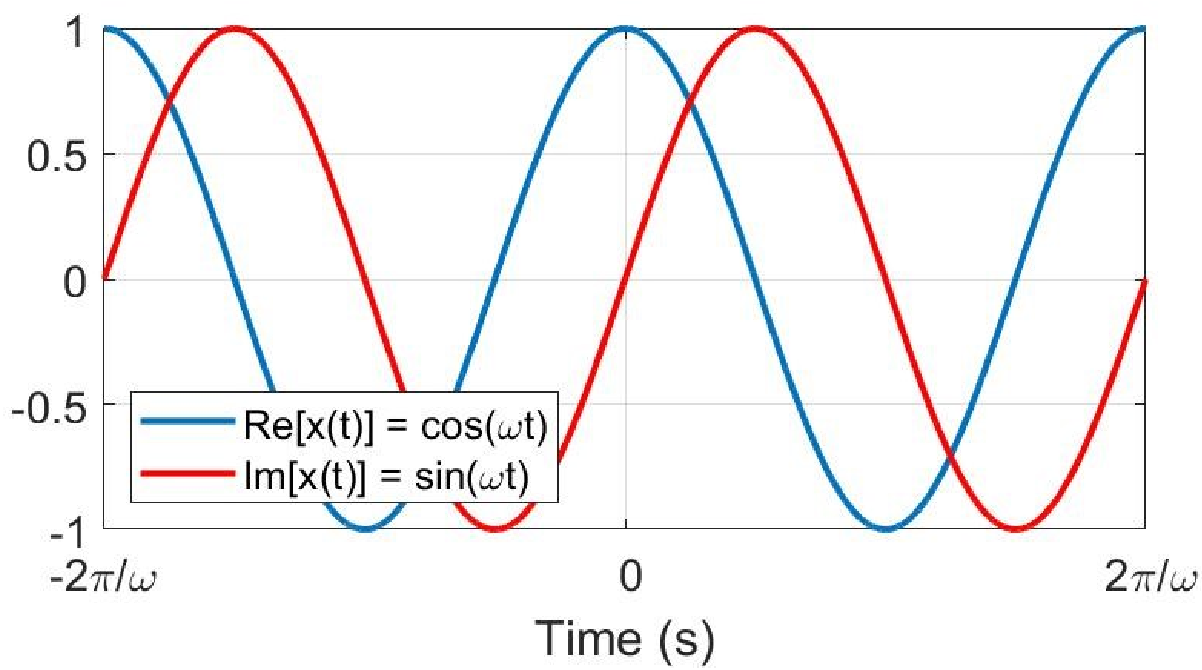
$$\left|e^{j\theta}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

(Blue circle has radius = 1;
The "unit circle")

- So, the expression $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$ is always on the unit circle of the complex plane for all values of time, and the angle ωt varies linearly with time at a rate of ω radians/second
- Positive frequency rotates counter-clockwise around the unit circle

Complex Sinusoid

- For $x(t) = \exp(j\omega t)$, the signal rotates around the unit circle;
  - real and imaginary parts vary with time, 90 degrees ($\pi/2$ rads) out of phase

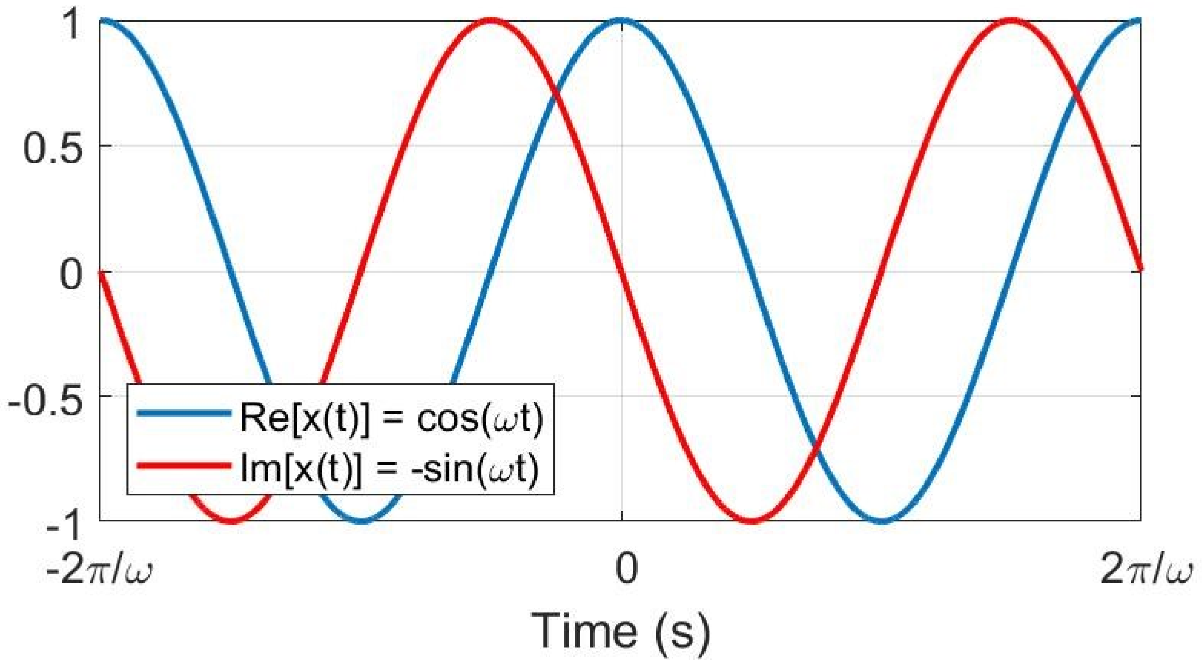


  - To travel once around the unit circle takes [2π radians]/[ω radians/sec]
    $= 2\pi/\omega = 1/F$ seconds

    ….where $F = \omega/2\pi$ is in units of cycles/second

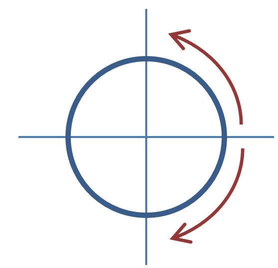
Complex Sinusoid

- The conjugate of the signal is $x(t) = \exp(-j\omega t) = \cos(\omega t) - j\sin(\omega t)$
  - Rotates in opposite direction around the unit circle



Real-Valued Sinusoids

- …are made up of oppositely rotating (conjugate) complex sinusoids, rotating at the same rate, such that the real or imaginary parts cancel for all time
- Cosine: $$\frac{e^{j\omega t}+e^{-j\omega t}}{2}=\frac{1}{2}\left(\cos(\omega t)+j\sin(\omega t)+\cos(\omega t)-j\sin(\omega t)\right)$$ $$=\frac{1}{2}\left(2\cos(\omega t)\right)=\cos(\omega t)=\cos(2\pi F t)$$ $$\omega = 2\pi F$$



  - add conjugates such that the imaginary parts cancel
- Sine: $$\frac{e^{j\omega t}-e^{-j\omega t}}{2j}=\frac{1}{2j}\left(\cos(\omega t)+j\sin(\omega t)-\cos(\omega t)+j\sin(\omega t)\right)$$ $$=\frac{1}{2j}\left(2j\sin(\omega t)\right)=\sin(\omega t)=\sin(2\pi F t)$$
  - subtract conjugates such that the real parts cancel, then divide by *j* to result in a real signal

CT Exponential Signals

- Now we generalize the exponential signal and allow the exponent to have "complex frequency"

$$x(t) = \exp(st) = \exp\big((\sigma + j\omega)t\big) = \underbrace{e^{\sigma t}}_{\substack{\text{Controls amplitude:}\\ \text{exponential growth}\\ \text{or decay}}} \; \underbrace{e^{j\omega t}}_{\substack{\text{Controls rotation around unit}\\ \text{circle (oscillating real \& imag}\\ \text{components)}}}$$

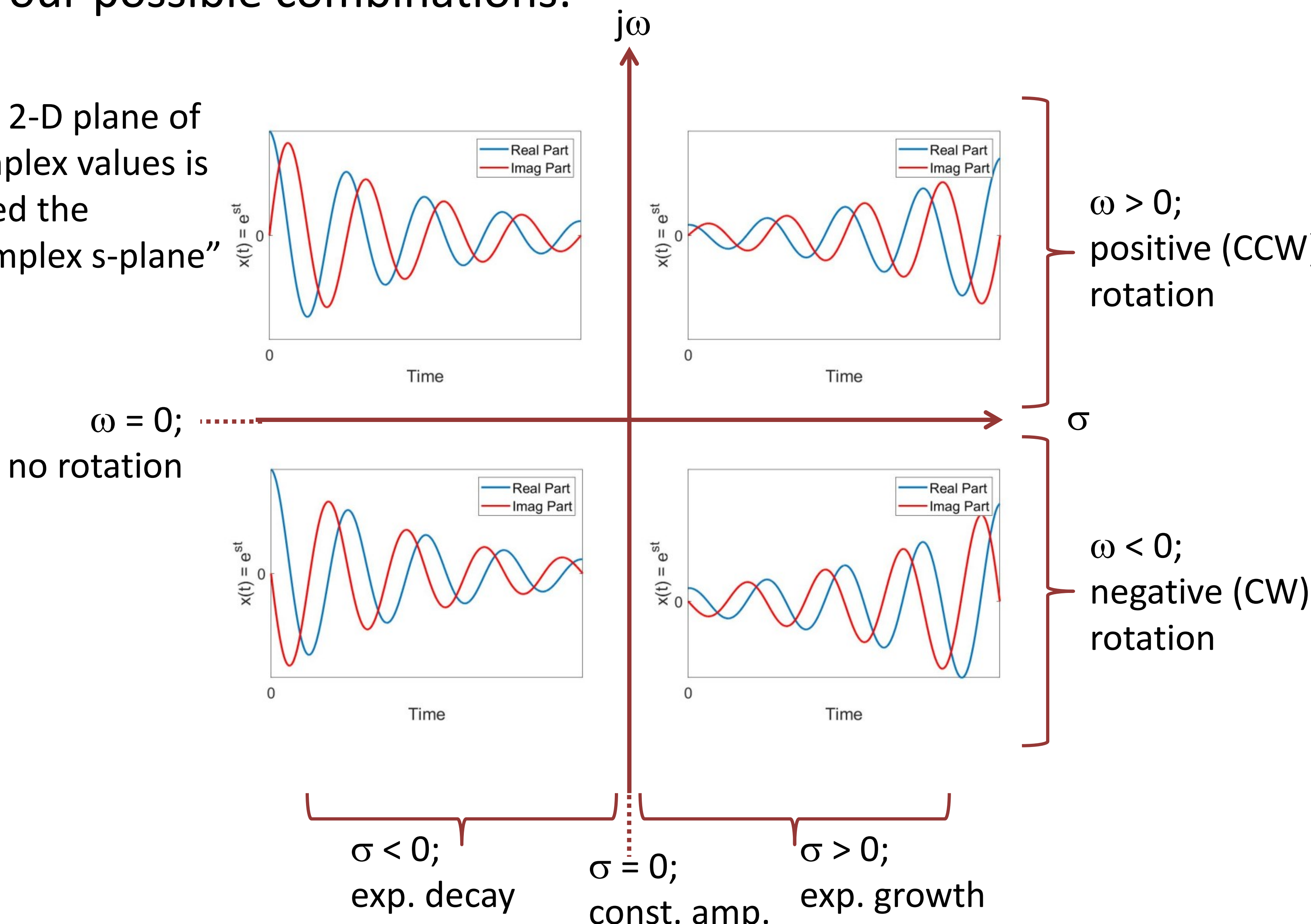
  – $s = \sigma + j\omega$ is called complex frequency
  – Different cases:
    - A) $s = 0$; then $e^{st} = 1$ (constant)
    - B) $s = \sigma$; then $e^{st} = e^{\sigma t}$; (real-valued exponentially growing or decaying)
    - C) $s = j\omega$; then $e^{st} = e^{j\omega t}$; (constant-amplitude, rotating phasor)
    - D) $s = \sigma + j\omega$; then $e^{st} = e^{\sigma t}e^{j\omega t} = e^{\sigma t}(\cos(\omega t) + j\sin(\omega t))$
      (rotating phasor with exponentially growing or decaying amplitude

    - $\sigma > 0 \rightarrow$ exponential growth as time increases
    - $\sigma < 0 \rightarrow$ exponential decay as time increases
    - $\sigma = 0 \rightarrow$ constant amplitude

CT Exponential Signals

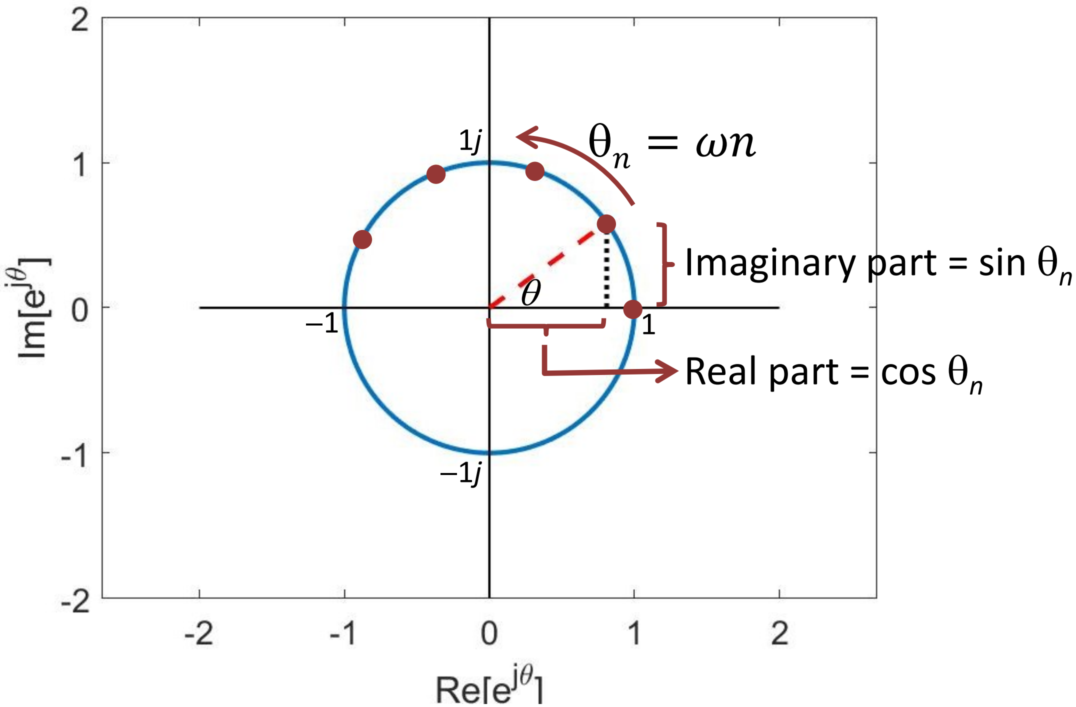
- Four possible combinations:

This 2-D plane of complex values is called the “complex s-plane”



DT Complex Sinusoid

$x[n] = \exp[j\omega n]$ *n* in "samples"; ω in radians/sample



- So, the expression $e^{j\omega n} = \cos(\omega n) + j\sin(\omega n)$ is always on the unit circle for all values of discrete time, and the angle "jumps" uniformly with time at ω radians per sample, continuing round and round the unit circle
  - (what happens if ω is greater than 2π?)
- Positive frequency rotates counter-clockwise around the unit circle

DT Exponential Signals

- The general DT exponential signal has the form: $x[n] = z^n$
  - where z is a possibly complex-valued number
  - Complex numbers can be expressed in magnitude and phase such that $z = \alpha e^{j\omega}$
  - Then…

$$x[n] = z^n = \left(\alpha e^{j\omega}\right)^n = \underbrace{\alpha^n}_{\text{Controls amplitude: exponential growth or decay}} \underbrace{e^{j\omega n}}_{\text{Controls rotation around unit circle (oscillating real \& imag components)}}$$

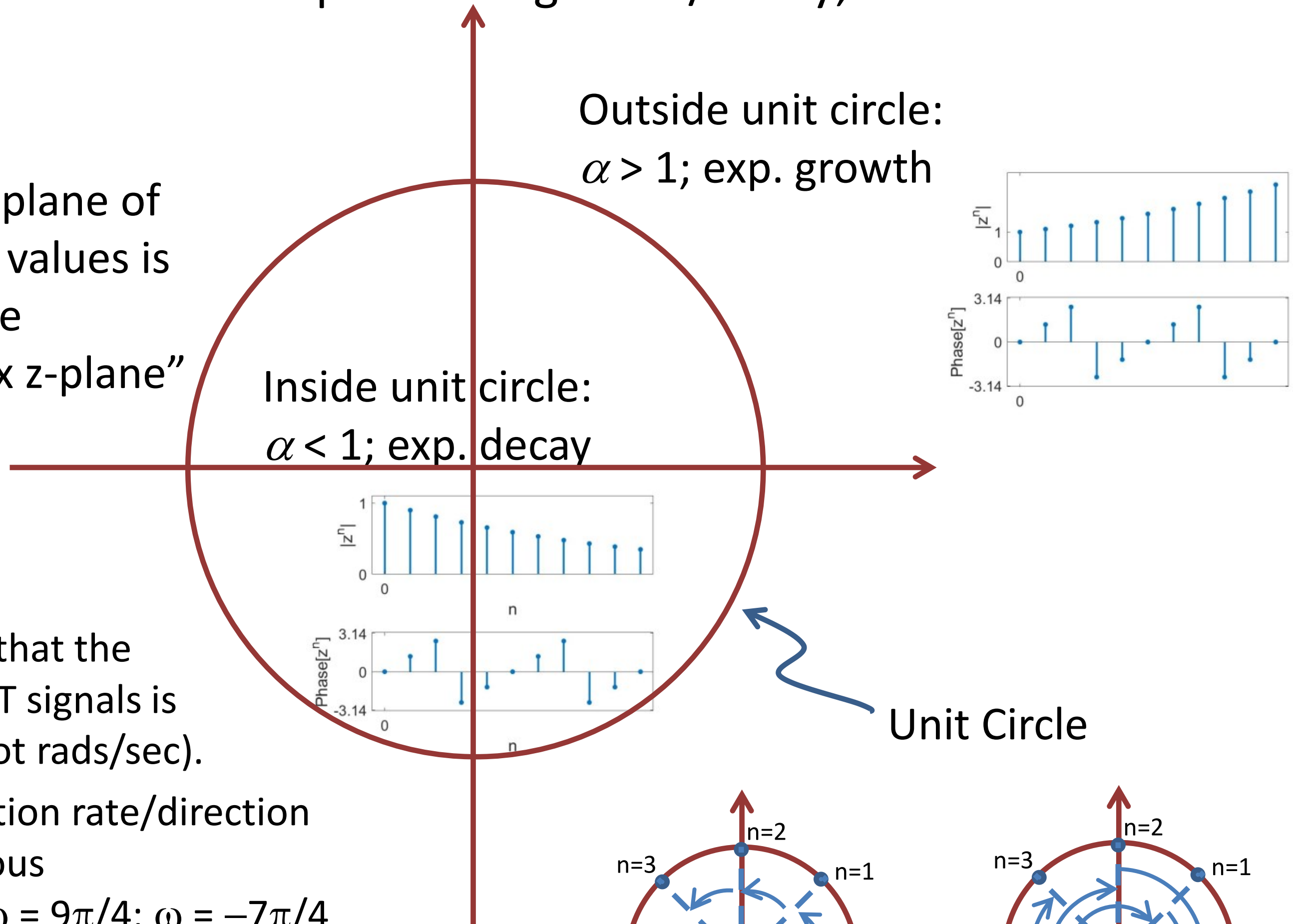
  - Different cases:
    - A) $z = 1$; then $z^n = 1$ (constant)
    - B) $\omega = 0$ (z is real); then $z^n = \alpha^n$; (real-valued exponentially growth or decay)
    - C) $\alpha = 1$ ($|z| = 1$); then $z^n = e^{j\omega n}$; (constant-amplitude, rotating phasor)
    - D) $z = \alpha e^{j\omega}$ (general z); $z^n = \alpha^n e^{j\omega n} = \alpha^n (\cos(\omega n) + j\sin(\omega n))$
      (rotating phasor with exponentially growing or decaying amplitude

    - $\alpha > 1 \rightarrow$ exponential growth as discrete time (n) increases
    - $\alpha < 1 \rightarrow$ exponential decay as discrete time (n) increases
    - $\alpha = 1 \rightarrow$ constant amplitude

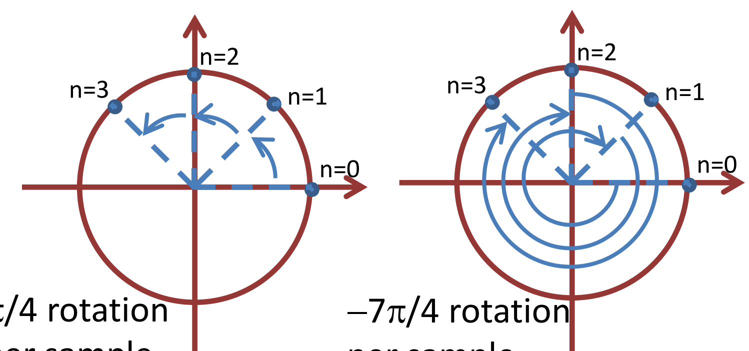
DT Exponential Signals

- Four possible combinations: exponential growth/decay; rotation direction

This 2-D plane of complex values is called the "complex z-plane"



  - However, note that the units of ω for DT signals is rads/sample (not rads/sec).
  - Therefore, rotation rate/direction can be ambiguous (e.g., $\omega = \pi/4$; $\omega = 9\pi/4$; $\omega = -7\pi/4$ all produce the same phases)
  - Because values are undefined in between, these rates are indistinguishable



$\pi/4$ rotation per sample

$-7\pi/4$ rotation per sample