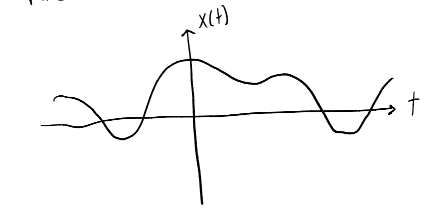
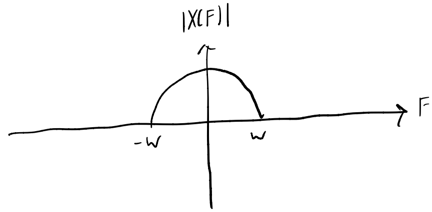
Sampling

The process by which we get from continuous waveforms to discrete ones

Time - Domain $x(t)$:

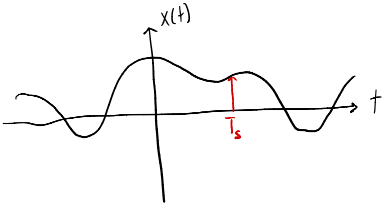


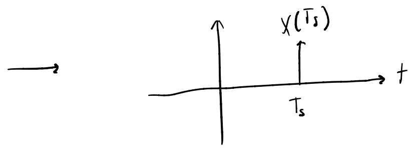
Frequency - Domain $X(F)$:



How to sample at a point $t = T_s$?

→ multiply by $\delta(t - T_s)$, then integrate (a dot product!)





sifting property

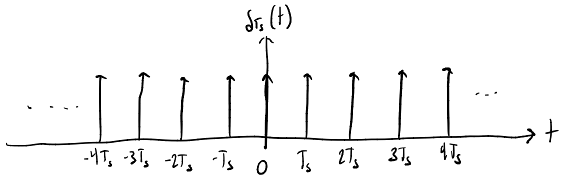
$$\longrightarrow x(t = T_s) = \langle x(t), \delta(t - T_s) \rangle = \int_{-\infty}^{\infty} x(t)\, \delta(t - T_s)\, dt = x(T_s)$$

Okay, but one sample isn't much information...

Let's multiply our function by a "train" of delta functions

$$\text{Let } \delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$T_s = 1/F_s$

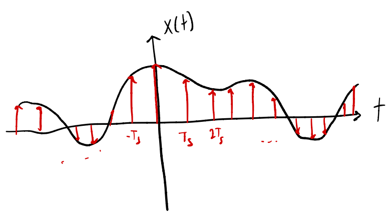


We get a "sampled sequence" by multiplying $x(t)$ by $\delta_{T_s}(t)$

$$\text{Let } \bar{x}(t) = x(t)\,\delta_{T_s}(t)$$

$$= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

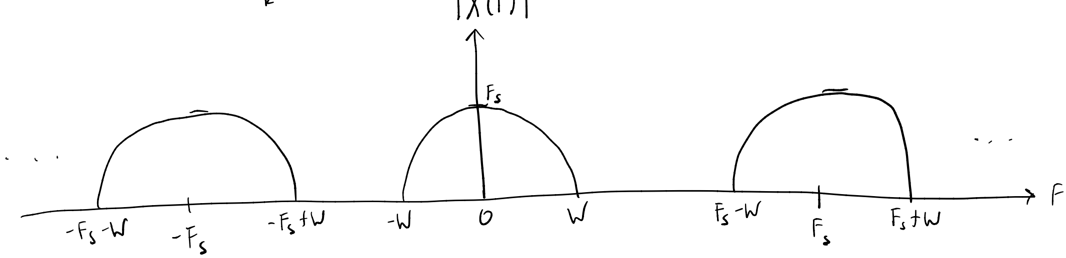
$$= \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s)$$



What does look like in the frequency domain?

- Multiplication in time is convolution in frequency
- So, $\bar{X}(F) = \underbrace{\mathcal{F}\{x(t)\}}_{= X(F)} * \underbrace{\mathcal{F}\{\delta_{T_s}(t)\}}_{= F_s \sum_{k=-\infty}^{\infty} \delta(F - kF_s)}$

$$\longrightarrow \bar{X}(F) = F_s \sum_{k=-\infty}^{\infty} X(F - kF_s)$$



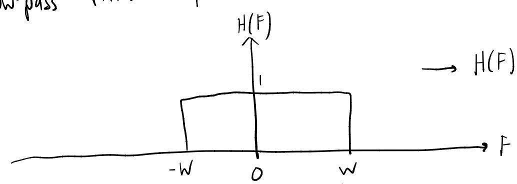
- Fourier transform becomes periodic! We have copies of original spectrum spaced out at integer multiples of sampling frequency and scaled in magnitude by F_s

$\bar{x}(t)$ is the "continuous" representation of the DT sequence $x[n]$

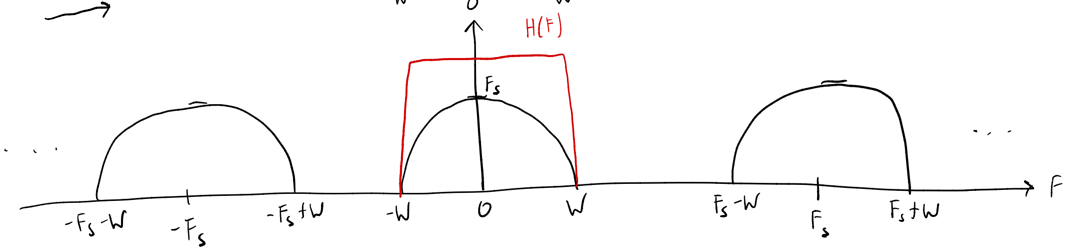
Q: Can we recover original $x(t)$ from $\bar{x}(t) / x[n]$?

A: Yes – we just need to filter out all the periodic copies of $X(F)$ from the spectrum of $\bar{x}(t)$

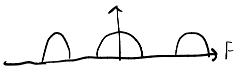
Define an ideal low-pass filter response $H(F)$:



$\longrightarrow H(F) = \mathrm{rect}\left(\frac{F}{2W}\right)$



$\longrightarrow$ Then $\bar{X}(F)H(F) = F_s X(F)$ and we get $\boxed{x(t) = \frac{1}{F_s}\mathcal{F}^{-1}\{\bar{X}(F)H(F)\}}$



$\bar{X}(F) \longrightarrow \boxed{\text{Ideal LPF}} \longrightarrow \propto X(F)$

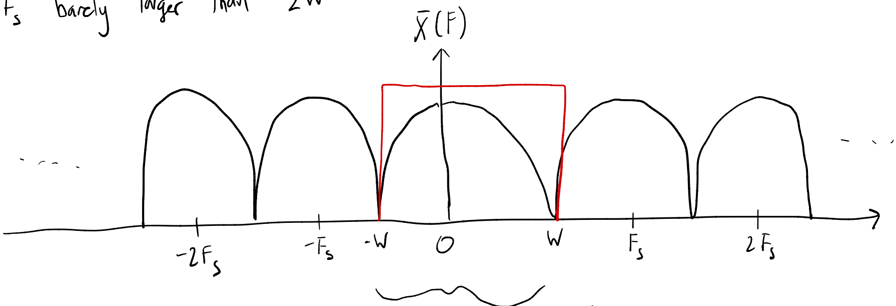
$H(F)$

What happens if we change F_s?

→ Copies of spectrum get closer or further apart

We can still recover original function if no overlap

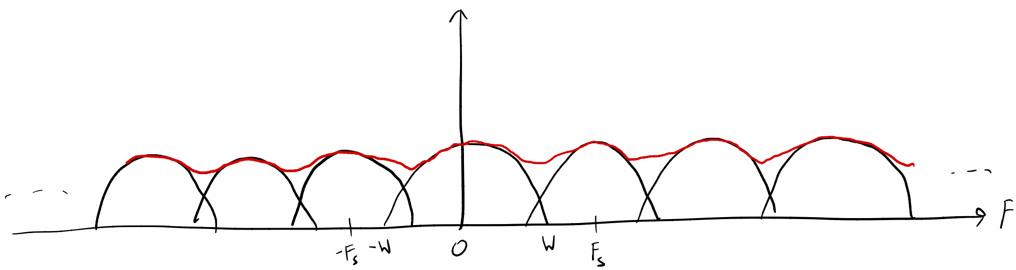
Ex. F_s barely larger than $2W$



Copies barely don't touch

– Theoretically, w/ ideal LPF, can still recover $X(t)$

What if $F_s < 2W$?



The periodic copies of $X(F)$ <u>overlap</u>.

After applying ideal LPF:

→ NOT equal to $X(F)$

The original signal cannot be recovered. The sampling rate is too slow, leading to spectral overlap and permanent distortion of the original signal.

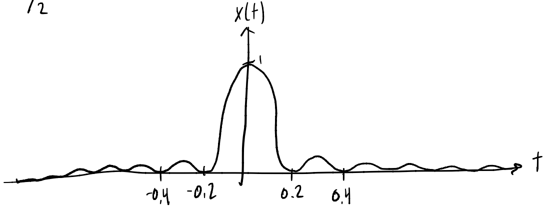
→ This overlap is called $\boxed{\text{aliasing.}}$

∴ A signal w/ max frequency W Hz ("band-limited" to W Hz) can be reconstructed exactly from its samples if the sampling frequency $\boxed{F_s > 2W}$

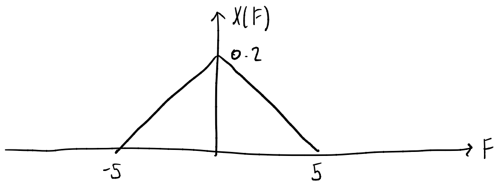
- The minimum sampling frequency $F_s = 2W$ is called the "Nyquist Rate" or the "Nyquist Frequency"

Another way: For a given set of samples taken at rate F_s then there is only one signal of bandwidth $B \le F_s/2$ that passes through those samples

Example: $x(t) = \text{sinc}^2(5t)$

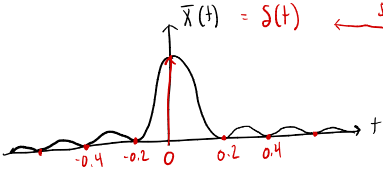


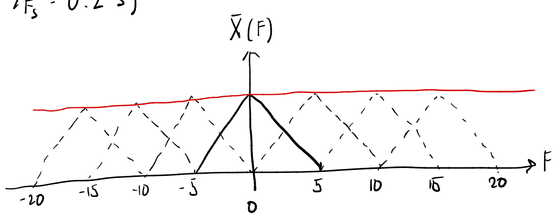
→ $X(F) = 0.2\,\Delta\left(\frac{f}{10}\right)$



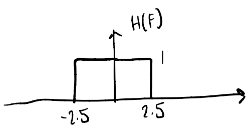
If we sample at $F_s = 5$ Hz ($T_s = 1/F_s = 0.2$ s)

$\bar{x}(t) = \delta(t) \xleftrightarrow{F} 1$

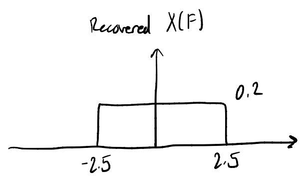




LPF:



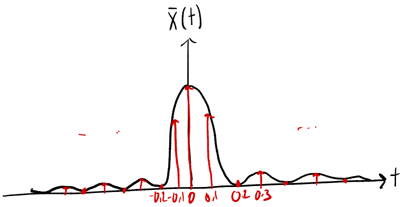
→

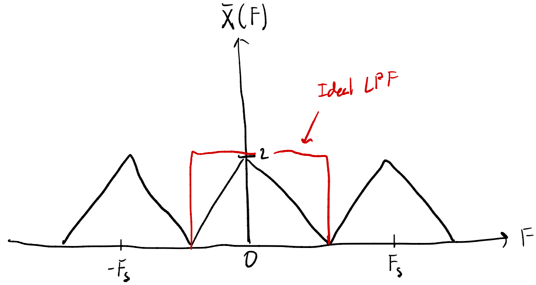


$= 0.2\,\text{rect}\left(\frac{F}{5}\right)$

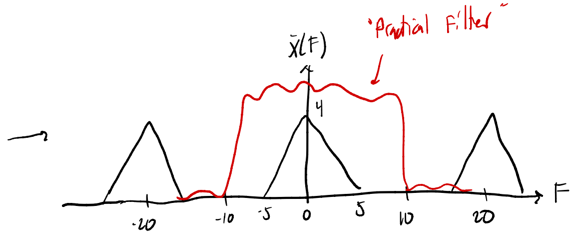
→ $x(t) = \text{sinc}(5t) \neq$ original $x(t)$

Nyquist Rate $F_s = 10$ Hz $(T_s = 0.1$ s$)$





Sample at $F_s = 20$ Hz:



Another perspective on aliasing:

$x_1(t) = \cos(2\pi F_1 t)$, $\quad x_2(t) = \cos(2\pi F_2 t)$

Suppose $F_2 = F_1 + kF_s$ for same F_s

(k integer)

$\rightarrow \bar{x}_1(nT_s) = \cos(2\pi F_1 n T_s)$

$$\bar{x}_2(nT_s) = \cos(2\pi(F_1 + kF_s)nT_s) = \cos(2\pi F_1 n T_s + 2\pi k n F_s T_s) = \cos(2\pi F_1 n T_s + \underbrace{2\pi k n}_{\text{int. multiple } 2\pi}) = \cos(2\pi F_1 n T_s) = \bar{x}_1(nT_s)$$

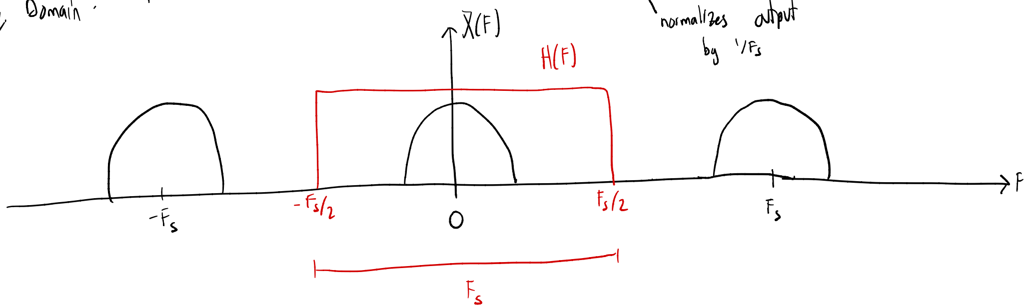
$\therefore$ Sinusoids spaced in frequency by integer multiples of F_s are indistinguishable from one another

$\rightarrow$ Matlab example

Reconstruction: Ok, we know how to sample - but how to reconstruct?

Freq. Domain: Filter out periodic replicas w/ $H(F) = \frac{1}{F_s} \text{rect}(F/F_s)$ (Assume signal band-limited below $F_s/2$)

normalizes output by $1/F_s$



Time - Domain:

$$H(F) = \frac{1}{F_s} \text{rect}(F/F_s) \xleftrightarrow{\mathcal{F}^{-1}} h(t) = \text{sinc}(F_s t)$$

$$X(F) = \bar{X}(F) H(F) \xleftrightarrow{\mathcal{F}^{-1}} x(t) = \bar{x}(t) * h(t)$$

$$= \left(\sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(t - nT_s) \right) * \text{sinc}(F_s t)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \left(\delta(t - nT_s) * \text{sinc}(F_s t) \right)$$

$$= \boxed{\sum_{n=-\infty}^{\infty} x(nT_s)\, \text{sinc}(F_s(t - nT_s))}$$

Ideal Interpolation Formula

$\therefore$ Original signal can be reconstructed w/ a weighted sum of sinc functions

Matlab example w/ $-5 \le t < 5$ and $-20 \le t < 20$

Infinite non-causal sinc not possible in reality... normally truncate or use a window

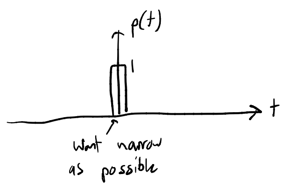
$\rightarrow h(t) = \text{sinc}(F_s t)\, w(t)$

$w(t) \propto$ 

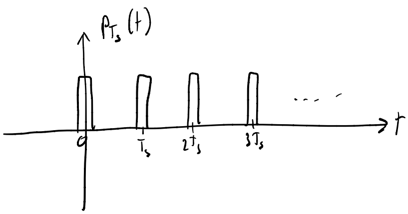
Practical Sampling

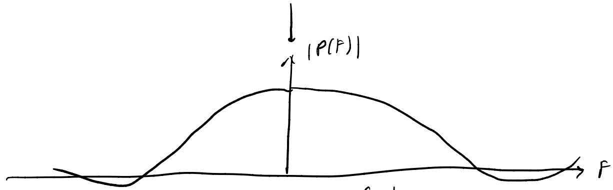
- In practice, we can't really make impulses ($\delta(t)$) for sampling ($\delta_{T_s}(t)$)
- Practical: A train of narrow pulses

$\rightarrow$ Suppose $p(t)$ is a single pulse

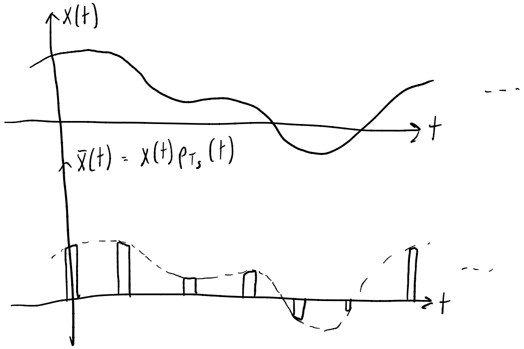


Pulse Train: $p_{T_s}(t) = \sum_{n=-\infty}^{\infty} p(t - nT_s)$





Very wide sinc function



What does $\bar{X}(F)$ look like?

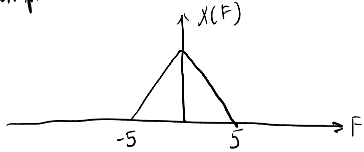
We get $p_{T_s}(t)$ by $p_{T_s}(t) = p(t) * \delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} p(t) * \delta(t - nT_s)$

$$\longrightarrow \bar{X}(F) = \mathcal{F}\{x(t)\, p_{T_s}(t)\} = \mathcal{F}\{x(t)\} * \mathcal{F}\{p_{T_s}(t)\} = X(F) * \mathcal{F}\{p_{T_s}(t)\}$$

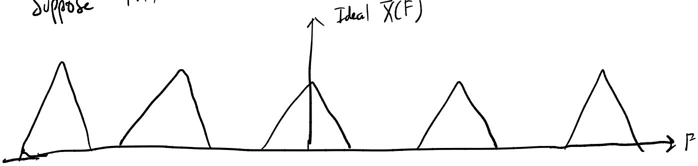
$$p_{T_s}(t) = p(t) * \delta_{T_s}(t) \xleftrightarrow{\mathcal{F}} P(F)\, F_s \sum_{k=-\infty}^{\infty} \delta(F - kF_s)$$

So, $\bar{X}(F) = X(F) * \left(P(F)\, F_s \sum_{k=-\infty}^{\infty} \delta(F - kF_s) \right) = P(F) \underbrace{F_s \sum_{k=-\infty}^{\infty} X(F - kF_s)}_{\text{Ideal } \bar{X}(F)} = P(F)\, \bar{X}_{ideal}(F)$

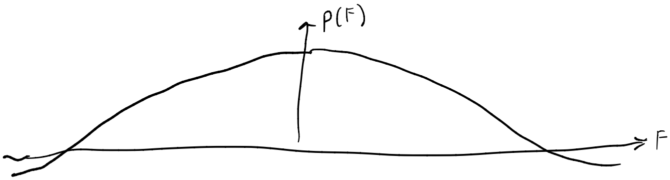
Previous Example: $x(t) = \text{sinc}^2(5t)$

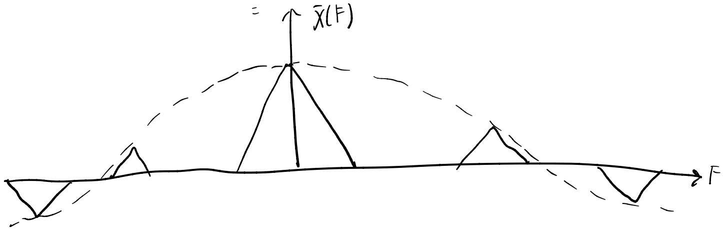


Suppose that width of $p(t) << T_s \longrightarrow$ BW of $p(t) >> F_s$



X





$\therefore$ As long as $P(F) \simeq$ constant within the interval $-W \leq F \leq W$ we can approximately reconstruct $x(t)$

In other words, $p(t)$ must be narrower than the rate of change of $x(t)$

As $p(t)$ gets narrower, $\longrightarrow \delta(t)$ and $P(F) \longrightarrow 1$ $\therefore$ ideal $\bar{X}(F)$ again