

The Laplace Transform

Problem: For some useful signals, the Fourier transform does not exist

• Example: $x(t) = e^t u(t)$

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_0^{\infty} e^t e^{-j\omega t} dt = \int_0^{\infty} e^{(1-j\omega)t} dt \\ &= \frac{e^{(1-j\omega)t}}{1-j\omega} \Big|_0^{\infty} = \infty \rightarrow \text{Does not converge, or DNE} \end{aligned}$$

Recall e^{st} : $s = \sigma + j\omega$

- Complex sinusoid w/ "complex frequency" w/ real & imaginary part
- More generalized than the $e^{j\omega t}$ used in Fourier

Laplace Transform:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$X(s) = \mathcal{L}\{x(t)\}$$

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) e^{st} ds$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$

Laplace transform has the same form as Fourier transform, but more generalized to include exponentials with a real part in the exponent's frequency

- May exist when the Fourier transform does not
- Particularly valuable for system analysis when FT does not exist, such as for unstable systems (more later)

Important: Like FT, Laplace is linear:

$$\mathcal{L}\{a_1 x_1(t) + a_2 x_2(t)\} = a_1 X_1(s) + a_2 X_2(s)$$

Region of Convergence:

Unlike the Fourier transform, which either exists or not, the Laplace transform has a region of s -values where it exists $\rightarrow$ Region of Convergence (ROC)

Example: $x(t) = e^{-at} u(t)$, a is real

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_0^{\infty} e^{-at} e^{-st} dt = \int_0^{\infty} e^{-(s+a)t} dt = \frac{e^{-(s+a)t}}{-(s+a)} \Big|_0^{\infty} = \boxed{\frac{1}{s+a}}$$

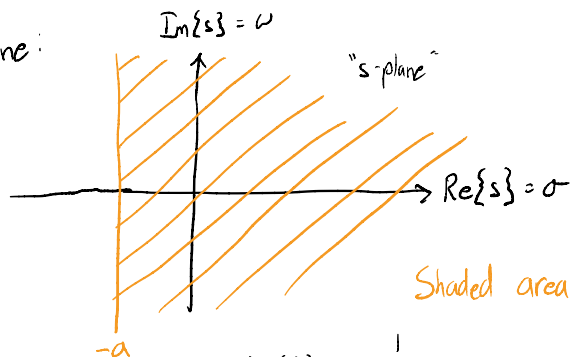
We assume that $\lim_{t \rightarrow \infty} e^{-(s+a)t} = 0 \rightarrow$ Only for some values of s !

If $\operatorname{Re}\{-s+a\} < 0$ or $\operatorname{Re}\{s+a\} > 0$, then $\lim_{t \rightarrow \infty} e^{-(s+a)t} = 0$ ✓

If $\operatorname{Re}\{-s+a\} > 0$ or $\operatorname{Re}\{s+a\} < 0$, then $\lim_{t \rightarrow \infty} e^{-(s+a)t} = \infty \rightarrow$ Diverges

So for the Laplace transform to exist, we need $\operatorname{Re}\{s+a\} > 0 \rightarrow \operatorname{Re}\{s\} > -a$
 $\rightarrow$ i.e., $X(s)$ exists for $x(t) = e^{-at} u(t)$ only for s where $\operatorname{Re}\{s\} > -a$

Often show in s -plane:



Shaded area is the ROC

If s is in the ROC, then $\mathcal{L}\{x(t)\} = \frac{1}{s+a}$

$$\rightarrow \mathcal{L}\{e^{-at} u(t)\} = \frac{1}{s+a}; \quad \text{ROC: } \operatorname{Re}\{s\} > -a$$

A ROC is required to uniquely determine $x(t)$ given $X(s)$

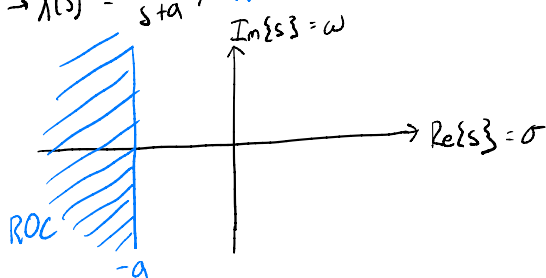
Example: $x(t) = -e^{-at}u(-t) \rightarrow X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^0 -e^{-at}e^{-st}dt$
 $= -\int_{-\infty}^0 e^{-(s+a)t}dt = -\left. \frac{e^{-(s+a)t}}{-(s+a)} \right|_{-\infty}^0 = \boxed{\frac{1}{s+a}}$

... But only if $\text{Re}\{-(s+a)\} > 0$ (so $e^{-(s+a)\infty} \rightarrow 0$)

$\text{Re}\{s\} < -a$ $\rightarrow$ Same $X(s)$ but different ROC! $\rightarrow$ Different signal

$$x(t) = -e^{-at}u(-t)$$

$$\rightarrow X(s) = \frac{1}{s+a}; \text{ ROC: } \text{Re}\{s\} < -a$$

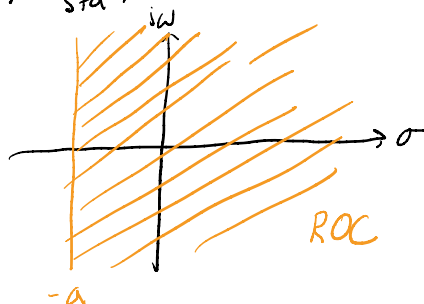


$x(t)$ is "Left-sided"

$x(t) = 0$ for $t > T_1$ (0 here)
 $\neq 0$ for $t < T_1$

$$x(t) = e^{-at}u(t)$$

$$\rightarrow X(s) = \frac{1}{s+a}; \text{ ROC: } \text{Re}\{s\} > -a$$



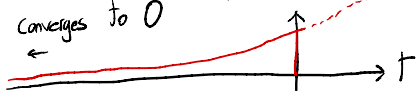
$x(t)$ is "right-sided"

$x(t) = 0$ for $t < T_1$ (0 here)
 $\neq 0$ for $t > T_1$

Left-Sided Signals:

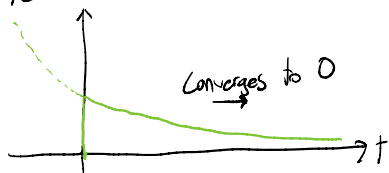
$$|e^{-(s+a)t}u(-t)|; \text{Re}\{s\} < -a$$

Converges to 0



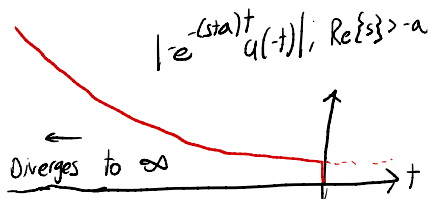
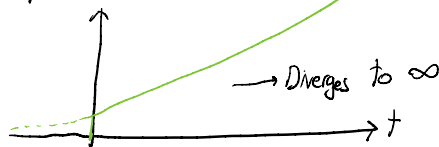
$$|e^{-(s+a)t}u(t)|; \text{Re}\{s\} > -a$$

Converges to 0



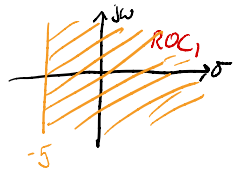
Right-Sided Signals:

$$|e^{-(s+a)t}u(t)|; \text{Re}\{s\} < -a$$

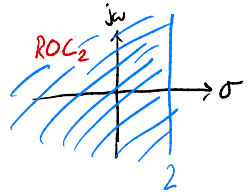


Example:

$$x_1(t) = e^{-5t} u(t) \xleftrightarrow{\mathcal{L}} X_1(s) = \frac{1}{s+5} ; \text{ROC: } \text{Re}\{s\} > -5$$



$$x_2(t) = -e^{2t} u(-t) \xleftrightarrow{\mathcal{L}} X_2(s) = \frac{1}{s-2} ; \text{ROC: } \text{Re}\{s\} < 2$$

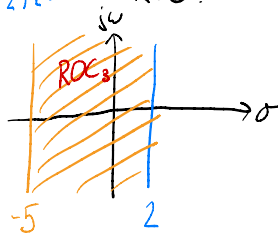


What about their sum?

$$\begin{aligned} \rightarrow x_3(t) = x_1(t) + x_2(t) &= \underbrace{e^{-5t} u(t)}_{\text{Right-Sided}} - \underbrace{e^{2t} u(-t)}_{\text{Left-Sided}} \xleftrightarrow{\mathcal{L}} X_3(s) = X_1(s) + X_2(s) \\ &= \frac{1}{s+5} + \frac{1}{s-2} \end{aligned}$$

"Two-Sided" Signal
Non-zero for $-\infty < t < \infty$

ROC? $\rightarrow$ Intersection of ROC_1 & ROC_2



$$\text{ROC: } -5 < \text{Re}\{s\} < 2$$

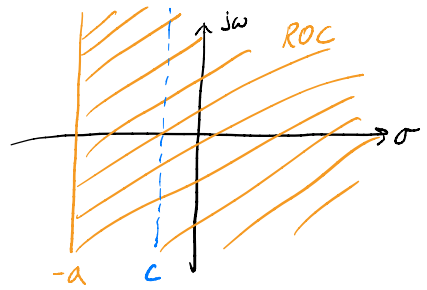
ROC Properties:

- Right-sided $x(t)$: ROC will contain all s where $\text{Re}\{s\} > \sigma_0$ for some σ_0
 $\rightarrow$ "Right half-plane"
- Left-sided $x(t)$: ROC will contain all s where $\text{Re}\{s\} < \sigma_0$ for some σ_0
 $\rightarrow$ "Left half-plane"
- Two-sided $x(t)$: Divide into right-sided $x_R(t)$ and left-sided $x_L(t)$
ROC is the overlap of right and left half-planes
No overlap? $\rightarrow$ ROC is empty & Laplace transform DNE
- Finite-duration $x(t)$: ROC is entire s -plane

Inverse Laplace Transform?

$$X(t) = \int_{c-j\infty}^{c+j\infty} X(s) e^{st} ds$$

↑ Pick a c -value in the ROC



Requires contour integration $\rightarrow$ use table instead

Laplace Transform Pairs and Properties:

Just like the FT, we can use table lookups and combine pairs & properties to solve Laplace transforms (or inverse Laplace)

Laplace has many similar pairs/properties to FT

e.g., $\mathcal{L}\{X(t-t_0)\} = X(s)e^{-st_0}$

$$\mathcal{L}\{X(at)\} = \frac{1}{|a|} X\left(\frac{s}{a}\right) \text{ etc...}$$

Note: Some modifications (like $X(at)$) require a change in ROC (also in table)

Laplace Transform Poles and Zeros

Very frequently, $X(s)$ can be written as a rational function of s :

$$X(s) = \frac{N(s)}{D(s)} \quad \begin{array}{l} \leftarrow \text{Polynomials in } s \\ \leftarrow \text{Polynomials in } s \end{array}$$

Why?

1. $X(s)$ is sum of exponentials (examples above)
2. We are using Laplace to solve a differential equation
e.g., RLC circuits, mechanical systems, etc.

If $N(s_0) = 0 \rightarrow X(s_0) = 0 \rightarrow s_0$ is called a zero of $X(s)$

If $D(s_0) = 0 \rightarrow X(s_0) \rightarrow \pm \infty \rightarrow s_0$ is called a pole of $X(s)$

Important: ROC of $X(s)$ can contain zeros but cannot contain poles

$N(s)$ and $D(s)$ can be factored:

$$X(s) = \frac{N(s)}{D(s)} = \frac{(s-b_1)(s-b_2)\dots(s-b_M)}{(s-a_1)(s-a_2)\dots(s-a_N)} = \frac{\prod_{i=1}^M (s-b_i)}{\prod_{j=1}^N (s-a_j)}$$

M zeros and N poles

If $O(N(s)) \neq O(D(s))$, there will be poles or zeros at $s = \infty$

- If $O(D(s)) > O(N(s))$ by k , there will be k zeros at ∞

- If $O(N(s)) > O(D(s))$ by k , there will be k poles at ∞

Inverse Laplace of rational $X(s)$?

Recall: $X(s) = \frac{1}{s+a} \xrightarrow{\mathcal{L}^{-1}} e^{-at} u(t)$ or $-e^{-at} u(-t)$

So $X(s) = \frac{A}{s+a_1} + \frac{B}{s+a_2} + \dots \xrightarrow{\mathcal{L}^{-1}} Ae^{-a_1 t} u(t) + Be^{-a_2 t} u(t) + \dots$

Get rational $X(s)$ into this form
using partial fraction expansion (PFE)

Most general PFE method:

$$A = (s+a_1) X(s) \Big|_{s=-a_1} = \underbrace{\frac{(s+a_1)A}{s+a_1}}_{= A} + \underbrace{\frac{(s+a_1)B}{s+a_2}}_{= 0} + \dots$$

$$B = (s+a_2) X(s) \Big|_{s=-a_2} = \underbrace{\frac{(s+a_2)A}{s+a_1}}_{= 0} + \underbrace{\frac{(s+a_2)B}{s+a_2}}_{= B} + \dots$$

Example:

$$X(s) = \frac{5s+11}{s^2+4s+3}$$

Find $x(t)$ for each possible ROC

Step 1: Factor $D(s)$ into $(s+a_1)(s+a_2)$ (Quadratic formula if necessary)

$$\rightarrow X(s) = \frac{5s+11}{(s+3)(s+1)} = \frac{A}{s+3} + \frac{B}{s+1}$$

Step 2: Solve for A and B

$$A = (s+3)X(s) \Big|_{s=-3} = \frac{5s+11}{s+1} \Big|_{s=-3} = \frac{-15+11}{-3+1} = \frac{-4}{-2} = \boxed{2}$$

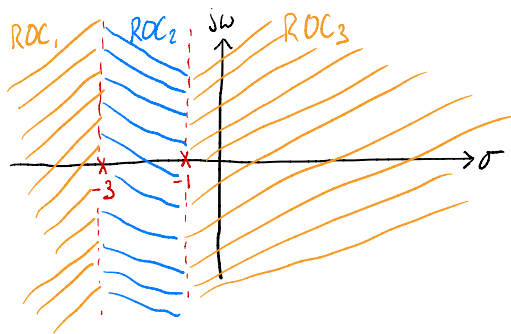
$$B = (s+1)X(s) \Big|_{s=-1} = \frac{5s+11}{s+3} \Big|_{s=-1} = \frac{-5+11}{-1+3} = \frac{6}{2} = \boxed{3}$$

Step 3: Determine possible ROCs

Recall: ROC cannot include a pole

Also, in general, ROCs are split up for different values of $\sigma = \text{Re}\{s\}$

$\rightarrow$ ROCs fall between real parts of poles and $\pm\infty$



ROC₁: $\text{Re}\{s\} < -3 \rightarrow$ Left-sided signal

ROC₂: $-3 < \text{Re}\{s\} < -1 \rightarrow$ Two-sided signal

ROC₃: $\text{Re}\{s\} > -1 \rightarrow$ Right-sided signal

Step 4: Find $x(t)$ for each ROC

• Each ROC produces a unique $x(t)$

- Each pole term $\frac{A}{s+a}$ produces an exponential/step-function
 - If the ROC is to the right of $-a \rightarrow Ae^{-at}u(t)$ Right-sided
 - If the ROC is to the left of $-a \rightarrow -Ae^{-at}u(-t)$ Left-sided

$$A=2, B=3$$

$$\text{ROC}_1: x(t) = -2e^{-3t}u(-t) - 3e^{-t}u(-t)$$

Left-sided

$$\text{ROC}_2: x(t) = 2e^{-3t}u(t) - 3e^{-t}u(-t)$$

Two-sided

$$\text{ROC}_3: x(t) = 2e^{-3t}u(t) + 3e^{-t}u(t)$$

Right-sided

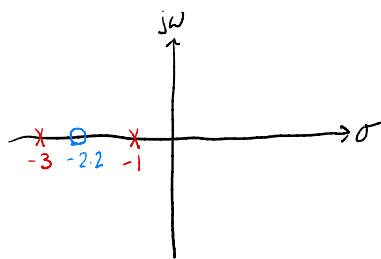
Pole-Zero Plots

Plot of all poles (marked by x's) and zeros (marked by o's) in s-plane

Previous example: $X(s) = \frac{5s+11}{(s+3)(s+1)}$

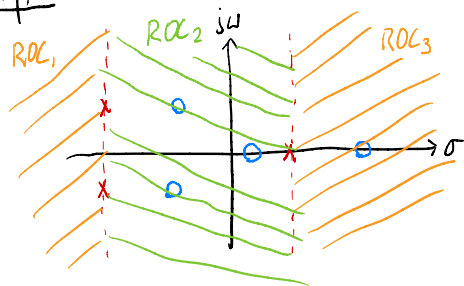
Poles: $s = -3, -1$

Zeros: $5s+11=0 \rightarrow s = -2.2$



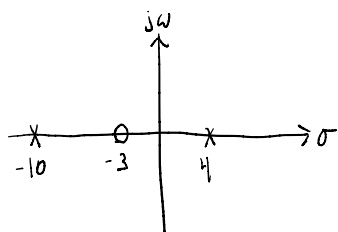
Remember: ROC of $X(s)$ can contain zeros but cannot contain poles

Example: Pole-zero plot of $X(s)$



If $X(s)$ is rational, we can determine it to within a scale factor by poles & zeros

Example:



$$N(s) = 0 \text{ at } s = -3 \rightarrow N(s) = K_N(s+3)$$

$$D(s) = 0 \text{ at } s = -10, +4 \rightarrow D(s) = K_D(s+10)(s-4)$$

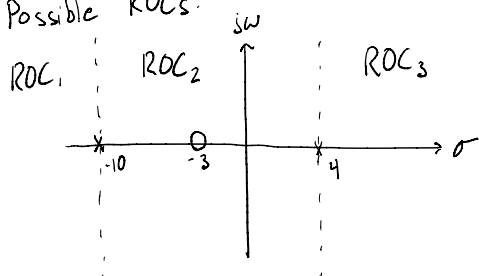
$$\rightarrow X(s) = \frac{K_N N(s)}{K_D D(s)} = K \frac{s+3}{(s+10)(s-4)} = K \frac{A}{s+10} + K \frac{B}{s-4}$$

$$A = X(s)(s+10) \Big|_{s=-10} = \frac{s+3}{s-4} \Big|_{s=-10} = \frac{-7}{-14} = \underline{\underline{\frac{1}{2}}}$$

$$B = X(s)(s-4) \Big|_{s=4} = \frac{s+3}{s+10} \Big|_{s=4} = \frac{7}{14} = \underline{\underline{\frac{1}{2}}}$$

$$\rightarrow X(s) = K \frac{0.5}{s+10} + K \frac{0.5}{s-4}$$

Possible ROCs:



$$ROC_1: \operatorname{Re}\{s\} < -10$$

$$ROC_2: -10 < \operatorname{Re}\{s\} < 4$$

$$ROC_3: \operatorname{Re}\{s\} > 4$$

$$ROC_1: \text{Left-sided signal} \rightarrow x(t) = K \left(-\frac{1}{2} e^{-10t} u(-t) - \frac{1}{2} e^{4t} u(-t) \right)$$

$$ROC_2: \text{Two-sided signal} \rightarrow x(t) = K \left(\frac{1}{2} e^{-10t} u(-t) - \frac{1}{2} e^{4t} u(-t) \right)$$

$$ROC_3: \text{Right-sided signal} \rightarrow x(t) = K \left(\frac{1}{2} e^{-10t} u(-t) + \frac{1}{2} e^{4t} u(t) \right)$$

Complex Poles?

$$X(s) = \frac{8}{s+2+j100} + \frac{8}{s+2-j100}$$

→ a is complex

→ ROC only concerned w/ real part of poles → i.e., $\text{Re}\{s\} > \text{Re}\{-a\}$

Suppose ROC: $\text{Re}\{s\} > -2$ → right-sided signal

$$\rightarrow X(t) = 8e^{-(2+j100)t} u(t) + 8e^{-(2-j100)t} u(t)$$

$$= 8e^{-2t} e^{-j100t} u(t) + 8e^{-2t} e^{j100t} u(t)$$

$$= 8e^{-2t} (e^{j100t} + e^{-j100t}) u(t) = 16e^{-2t} \cos(100t) u(t)$$

→ $X(t)$ can only be real-valued if poles are purely real or come in complex conjugate pairs

- $X(s)$ must be conjugate symmetric across the $j\omega=0$ axis
(just like how the FT must be cons. sym. to have real $x(t)$)

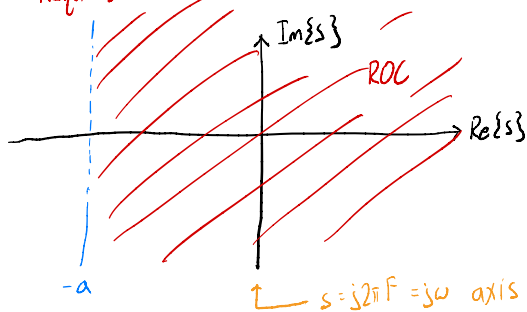
Relationship to Fourier Transform

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$X(F) = X(s) \Big|_{s=j\omega=j2\pi F} = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt$$

Most of the time, $X(F)$ can be found by $X(s) \Big|_{s=j2\pi F}$

→ Requires that $s=j2\pi F$ be in the ROC!



Evaluating $X(s)$ along this axis
gives $X(F)$ or $X(\omega)$

Another way to think about Laplace transform:

- Consider causal signals - signals that turn "on" at $t=0$

★ Very important for Laplace! Some useful Laplace properties only apply to causal signals → "Unilateral Laplace Transform"

Causal $x(t)$:

$$X(\omega) = \int_0^{\infty} x(t) e^{-j\omega t} dt$$

$$X(s) = X(\sigma + j\omega) = \int_0^{\infty} x(t) e^{-st} dt = \int_0^{\infty} \underbrace{x(t) e^{-\sigma t}}_{\substack{\text{Fourier transform} \\ \text{of } x(t) e^{-\sigma t}}} e^{-j\omega t} dt$$

$e^{-\sigma t}$ is a "convergence factor" that helps the Fourier transform exist when it otherwise would not for some values of σ

- $X(s)$ exists so long as $x(t)$ grows slower than $e^{-\sigma t}$ decays ($\text{Re}\{s\} = \sigma$ in ROC!)
- If we set $\sigma=0$, we get original Fourier transform

$$\text{Let } x(t) = e^{-at} u(t) \rightarrow X(s) = \frac{1}{s+a}; \text{ ROC: } \sigma = \text{Re}\{s\} > -a$$

- If $a > 0$, $x(t)$ is decaying, and $e^{-\sigma t}$ can be allowed to grow slightly as long as it grows slower than $x(t)$ decays $\rightarrow x(t)e^{-\sigma t}$ still decays $\rightarrow \underline{X(s) \text{ exists}}$
- If $a < 0$, then $x(t)$ grows exponentially, so $e^{-\sigma t}$ must decay faster than $x(t)$ grows to counteract $\rightarrow x(t)e^{-\sigma t}$ still decays $\rightarrow \underline{X(s) \text{ exists}}$

Properties of the Laplace Transform:

Convolution property: $y(t) = x(t) * h(t) \longleftrightarrow Y(s) = \underbrace{X(s)}_{\text{Output}} \underbrace{H(s)}_{\text{Input}} \underbrace{H(s)}_{\text{System "Transfer Function"}}$

$H(s) = \mathcal{L}\{h(t)\}$

For unilateral Laplace transform: $X(s) = \int_0^{\infty} x(t) e^{-st} dt$

Initial value theorem: $x(0^+) = \lim_{s \rightarrow \infty} sX(s)$

Final value theorem: $\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s)$

Time differentiation property:

$$\frac{dx(t)}{dt} = sX(s) - \underbrace{x(0^-)}_{\text{Initial condition (IC) - Important in diff. eq.}}$$

$$\frac{d^2x(t)}{dt^2} = s^2X(s) - sx(0^-) - x'(0^-)$$

$$\vdots$$
$$\frac{d^n x(t)}{dt^n} = s^n X(s) - \sum_{k=1}^n s^{n-k} \underbrace{x^{(k-1)}}_{\text{Refers to } (k-1)^{\text{th}} \text{ derivative of } x}(0^-)$$

Solving LCCDEs

Time-differentiation property makes Laplace useful for solving differential equations

Linear Constant Coefficient Differential Equations (LCCDEs)

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_0 y(t) = b_M \frac{d^M x(t)}{dt^M} + \dots + b_0 x(t)$$

$$\rightarrow \sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}$$

$y(t)$ output, $x(t)$ input $\rightarrow$ LCCDEs describe systems!

$\rightarrow$ Any system described by an LCCDE is LTI

• e.g., RLC circuits, mechanical systems, optics, etc.

In this class, there are 4 main things we might want to solve an LCCDE for:

1. Determine system output $y(t)$ due to an input signal $x(t)$

$\rightarrow$ "zero-state solution"

2. Determine system output $y(t)$ due to system initial conditions

$\rightarrow$ "zero-input solution"

3. Determine system output $y(t)$ due to input signal and IC's

$\rightarrow$ "total solution"

4. Determine system transfer function $H(s)$ & impulse response $h(t)$

Describe
output signal

Describes system

Example:
$$\frac{d^2 y(t)}{dt^2} + 5 \frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + x(t)$$

Initial conditions: $y(0^-) = 2$ $\dot{y}(0^-) = 1$ (Note: we need N IC's)

Input: $x(t) = e^{-4t}u(t)$ (Note: we will assume $x(t)$ IC's = 0)

Recall: $\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{L}} sX(s) - x(0^-)$ and $\frac{d^2 x(t)}{dt^2} \xleftrightarrow{\mathcal{L}} s^2 X(s) - sx(0^-) - \dot{x}(0^-)$

$\rightarrow$ Apply the Laplace transform to both sides of equation to get expression in s , $Y(s)$, and $X(s)$

$$\rightarrow \frac{d^2 y(t)}{dt^2} \xleftrightarrow{\mathcal{L}} s^2 Y(s) - sy(0^-) - \dot{y}(0^-) = s^2 Y(s) - 2s - 1$$

$$5 \frac{dy(t)}{dt} \xleftrightarrow{\mathcal{L}} 5(sY(s) - y(0^-)) = 5sY(s) - 5 \cdot 2 = 5sY(s) - 10$$

$$6y(t) \xleftrightarrow{\mathcal{L}} 6Y(s)$$

$$\rightarrow \text{LHS} = s^2 Y(s) - 2s - 1 + 5sY(s) - 10 + 6Y(s) = \underline{(s^2 + 5s + 6)Y(s) - 2s - 11}$$

Now right-hand side...

$$\frac{dx(t)}{dt} \xrightarrow{\mathcal{L}} sX(s) + \underbrace{x(0^-)}_{=0} = sX(s)$$

$$x(t) \xrightarrow{\mathcal{L}} X(s)$$

$$x(t) = e^{-4t} u(t) \xleftrightarrow{\mathcal{L}} X(s) = \frac{1}{s+4}$$

$$\rightarrow \text{RHS} = sX(s) + X(s) = (s+1)X(s) = \frac{s+1}{s+4}$$

$$\rightarrow (s^2 + 5s + 6)Y(s) - 2s - 11 = \frac{s+1}{s+4} \quad \left. \vphantom{\frac{s+1}{s+4}} \right\} \text{LCDE converted to algebraic eq. of } s$$

→ Now, need to solve for $Y(s)$ to get to $y(t)$

$$\rightarrow (s^2 + 5s + 6)Y(s) = \frac{s+1}{s+4} + 2s + 11 = \frac{s+1 + (s+4)(2s+11)}{s+4} = \frac{s+1 + 2s^2 + 8s + 11s + 44}{s+4} = \frac{2s^2 + 20s + 45}{s+4}$$

$$\rightarrow Y(s) = \frac{2s^2 + 20s + 45}{(s+4)(s^2 + 5s + 6)} = \frac{2s^2 + 20s + 45}{(s+4)(s+3)(s+2)} = \frac{A}{s+4} + \frac{B}{s+3} + \frac{C}{s+2}$$

$$A = (s+4)Y(s) \Big|_{s=-4} = \frac{2 \cdot (-4) - 4 \cdot 20 + 45}{(-4+3)(-4+2)} = \frac{32 - 80 + 45}{(-1)(-2)} = -\frac{3}{2}$$

$$B = (s+3)Y(s) \Big|_{s=-3} = \frac{2 \cdot (-3) - 3 \cdot 20 + 45}{(-3+4)(-3+2)} = \frac{18 - 60 + 45}{(1)(-1)} = -3$$

$$C = (s+2)Y(s) \Big|_{s=-2} = \frac{2 \cdot (-2) - 2 \cdot 20 + 45}{(-2+4)(-2+3)} = \frac{8 - 40 + 45}{(2)(1)} = \frac{13}{2}$$

$$\rightarrow Y(s) = -\frac{3/2}{s+4} - \frac{3}{s+4} + \frac{13/2}{s+2} \xleftrightarrow{\mathcal{L}} y(t) = \left(\frac{13}{2} e^{-2t} - 3e^{-3t} - \frac{3}{2} e^{-4t} \right) u(t)$$

* Note: $y(t)$ needs to be right-sided because IC's use unilateral Laplace $\rightarrow$ causal
Will not always be the case for $h(t)$!

The above solution includes input and IC's $\rightarrow$ Total Solution

"Zero-state solution": $y_{zs}(t)$, due to input only $\rightarrow$ Set IC's = 0 and solve

"Zero-input solution": $y_{zi}(t)$, due to IC's only $\rightarrow$ Set $x(t)/X(s) = 0$ and solve

Total solution $y(t) = y_{zs}(t) + y_{zi}(t)$

Zero-State Solution $y_{zs}(t)$:

Previous example, but set $y(0) = \dot{y}(0) = 0$

$$\rightarrow (s^2 + 5s + 6)Y(s) = \frac{s+1}{s+4} \rightarrow Y(s) = \frac{s+1}{(s+4)(s+3)(s+2)} = \frac{A}{s+4} + \frac{B}{s+3} + \frac{C}{s+2}$$

$$A = (s+4)Y(s)|_{s=-4} = \frac{-4+1}{(-4+3)(-4+2)} = \frac{-3}{(-1)(-2)} = -\frac{3}{2}$$

$$B = (s+3)Y(s)|_{s=-3} = \frac{-3+1}{(-3+4)(-3+2)} = \frac{-2}{(1)(-1)} = 2$$

$$C = (s+2)Y(s)|_{s=-2} = \frac{-2+1}{(-2+4)(-2+3)} = \frac{-1}{(2)(1)} = -\frac{1}{2}$$

$$\rightarrow Y(s) = -\frac{3/2}{s+4} + \frac{2}{s+3} - \frac{1/2}{s+2} \xleftrightarrow{\mathcal{L}^{-1}} y_{zs}(t) = \left(-\frac{3}{2}e^{-4t} + 2e^{-3t} - \frac{1}{2}e^{-2t} \right) u(t)$$

* Like above, generally assume output signals will be causal

Zero-Input Solution $y_{zi}(t)$:

Set $x(t) = 0 \xleftrightarrow{\mathcal{L}} X(s) = 0$

$$\rightarrow (s^2 + 5s + 6)Y(s) - 2s - 11 = 0$$

$$\rightarrow (s^2 + 5s + 6)Y(s) = 2s + 11 \rightarrow Y(s) = \frac{2s+11}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2}$$

$$A = (s+3)Y(s)|_{s=-3} = \frac{-3 \cdot 2 + 11}{-3+2} = \frac{-6+11}{-1} = -5$$

$$B = (s+2)Y(s)|_{s=-2} = \frac{-2 \cdot 2 + 11}{-2+3} = \frac{-4+11}{1} = 7$$

$$Y(s) = -\frac{5}{s+3} + \frac{7}{s+2} \xleftrightarrow{\mathcal{L}^{-1}} y_{zi}(t) = \left(-5e^{-3t} + 7e^{-2t} \right) u(t)$$

Could also solve for total solution from here: $y(t) = y_{zs}(t) + y_{zi}(t)$

Impulse Response:

Remember, LCCDEs describe LTI systems $\rightarrow y(t) = x(t) * h(t) \xleftrightarrow{\text{FT}} Y(s) = X(s) \underline{H(s)}$

$H(s)$ is system transfer function, $H(s) \xleftrightarrow{\text{FT}} h(t)$ impulse response

To find $H(s)/h(t)$, set ICs = 0 and $x(t) = \delta(t) \xleftrightarrow{\text{FT}} \underline{X(s) = 1}$; ROC: all s

Back to example:

$$(s^2 + 5s + 6)Y(s) - \underbrace{2s - 11}_{\text{IC's}} = (s+1)X(s)$$

$$\rightarrow \text{Set IC's} = 0 \text{ and } X(s) = 1$$

$$\rightarrow (s^2 + 5s + 6)Y(s) = s + 1 \rightarrow Y(s) = \frac{s+1}{s^2 + 5s + 6}$$

$$\text{From above: } Y(s) = X(s)H(s) \rightarrow H(s) = \frac{Y(s)}{X(s)} \quad \text{but } X(s) = 1, \text{ so } H(s) = Y(s) \text{ here}$$

$$\rightarrow H(s) = Y(s) = \frac{s+1}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2}$$

$$A = (s+3)H(s)|_{s=-3} = \frac{-3+1}{-3+2} = \frac{-2}{-1} = \underline{2}$$

$$B = (s+2)H(s)|_{s=-2} = \frac{-2+1}{-2+3} = \frac{-1}{1} = \underline{-1}$$

$$\rightarrow H(s) = \frac{2}{s+3} - \frac{1}{s+2} \quad \text{Poles at } s = -2, -3$$

Unlike $y(t)$, $h(t)$ is not required to be right-sided! We need info about the system to determine ROC and $h(t)$.

Often though, we assume system is causal (i.e., $h(t) = 0$ for $t < 0$), so $h(t)$ right-sided

$$\rightarrow \boxed{h(t) = (2e^{-3t} - e^{-2t})u(t)}$$

LCCDE Summary:

Given a system defined by an LCCDE: (you will also know IC's and $x(t)$)

1.) Use differentiation property of Laplace to get

$$\underbrace{Q(s)}_{\text{From } y\text{-derivative coefficients}} Y(s) + \underbrace{f(s)}_{\text{From IC's}} = \underbrace{P(s)}_{\text{From } x\text{-derivative coefficients}} X(s)$$

2.) For total solution $y(t)$

- Rearrange to $Y(s) = \frac{P(s)X(s) - f(s)}{Q(s)}$

- Do PFE then inverse Laplace

3.) For "zero-state" solution $y_{zs}(t)$

- Same as #2 but set IC's = 0 (i.e., $f(s) = 0$)

4.) For "zero-input" solution $y_{zi}(t)$

- Same as #2 but set $X(s) = 0$

5.) For impulse response of system

- Same as #2 but set IC's = 0 ($f(s) = 0$) and $X(s) = 1$ (b/c $x(t) = \delta(t)$)

System Analysis w/ Laplace Transform

$$\text{Recall } \frac{d^n y(t)}{dt^n} = s^n Y(s) - \sum_{k=1}^n s^{n-k} y^{(k-1)}(0^-)$$

For convolution / transfer function, we don't need IC's $\rightarrow \frac{d^n y(t)}{dt^n} = s^n Y(s)$

$$\text{LCODE: } a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_0 y(t) = b_M \frac{d^M x(t)}{dt^M} + \dots + b_0 x(t)$$

$$\xleftrightarrow{\mathcal{L}} a_N s^N Y(s) + a_{N-1} s^{N-1} Y(s) + \dots + a_0 Y(s) = b_M s^M X(s) + \dots + b_0 X(s)$$

$$\rightarrow (a_N s^N + a_{N-1} s^{N-1} + \dots + a_0) Y(s) = (b_M s^M + \dots + b_0) X(s)$$

$$\rightarrow Y(s) \sum_{k=0}^N a_k s^k = X(s) \sum_{k=0}^M b_k s^k$$

$$\rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^M b_k s^k}{\sum_{k=0}^N a_k s^k}$$

Polynomial of $s \rightarrow$ gives zeros

Rational $H(s)$

Polynomial of $s \rightarrow$ gives poles

$$y(t) = x(t) * h(t) \xleftrightarrow{\mathcal{L}} Y(s) = X(s) H(s)$$

If $j\omega$ (or $j2\pi F$) axis in ROC, then

$$y(t) = x(t) * h(t) \xleftrightarrow{\mathcal{F}} Y(\omega) = H(\omega) X(\omega) \quad \text{or} \quad Y(F) = H(F) X(F)$$

Problem: ROC of $H(s)$ not always right-sided! We need some extra information to determine the ROC and $h(t)$

We will focus on 2 key system descriptors: causality and stability

Causality:

Output cannot respond to future input $\rightarrow h(t) = 0$ for $t < 0$

$\rightarrow h(t)$ must be right-sided!

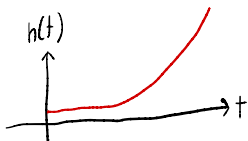
$\rightarrow$ ROC of $H(s)$ is a right half-plane

If $H(s)$ is rational, the system is causal only if ROC is to the right of the right-most pole.

(Note: the converse is not necessarily true. ROC can be right half-plane and $h(t)$ can be right-sided but still turn on at $t < 0$.)

BIBO Stability:

What is "unstable"? e.g.,



$\rightarrow$ A bounded input (impulse) produces an "unbounded" output that diverges to ∞

$\rightarrow$ Usually a bad thing in system design

Stability criterion: $h(t)$ is "absolutely integrable" $\rightarrow \int_{-\infty}^{\infty} |h(t)| dt < \infty$

$\rightarrow$ Implies existence/convergence of FT $H(\omega)/H(f)$

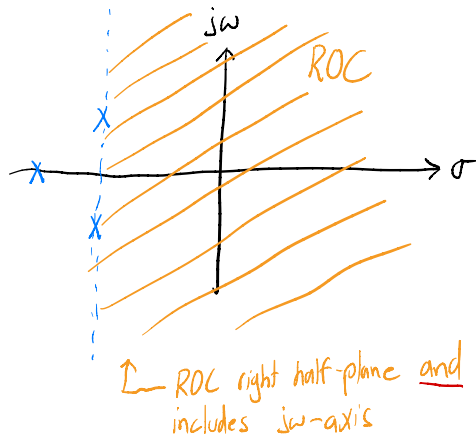
$\rightarrow s = j\omega$ axis must be in the ROC

An LTI system is stable if and only if the ROC of $H(s)$ includes the $j\omega$ -axis

Causal $\rightarrow$ ROC is a right half-plane

Stable $\rightarrow$ ROC includes the $j\omega$ -axis

$\rightarrow$ For a causal LTI system w/ rational $H(s)$, the system is stable if and only if all poles are in the left half of the s -plane (i.e., real part of all poles < 0)



$\rightarrow$ ROC right half-plane and includes $j\omega$ -axis

Example: $H(s) = \frac{2s+1}{s^2+s-6} = \frac{2s+1}{(s+3)(s-2)} = \frac{A}{s+3} + \frac{B}{s-2}$

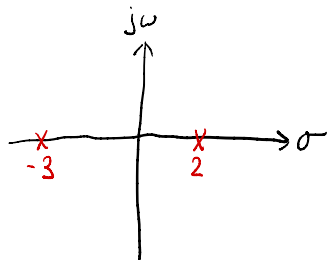
$$A = H(s)(s+3) \Big|_{s=-3} = \frac{-6+1}{-3-2} = \underline{1}$$

$$B = H(s)(s-2) \Big|_{s=2} = \frac{4+1}{2+3} = \underline{1}$$

$$\rightarrow H(s) = \frac{1}{s+3} + \frac{1}{s-2}$$

Can system be both causal & stable?

No, because there is a pole to the right of the $j\omega$ -axis



$h(t)$ if system is causal? $\rightarrow \text{ROC: } \text{Re}\{s\} > 2 \rightarrow h(t) = e^{-3t}u(t) + e^{2t}u(t)$

converges diverges

Both are causal

$h(t)$ if system is stable? $\rightarrow \text{ROC: } -3 < \text{Re}\{s\} < 2 \rightarrow h(t) = e^{-3t}u(t) - e^{2t}u(-t)$

causal anti-causal

Both converge

Response of Systems to Sinusoidal Inputs

Recall eigenfunction property of LTI systems:

$$x(t) = e^{j\omega_0 t} \rightarrow \boxed{H(\omega)} \rightarrow y(t) = H(\omega_0) e^{j\omega_0 t}$$

→ Input sinusoid of freq. ω_0 , output has same freq. but scaled by complex scalar $H(\omega_0)$

→ $H(\omega_0)$ describes how system responds to sinusoid of freq. ω_0

Could apply FT, but only for $s = j\omega_0$

But it turns out the eigenfunction property works for e^{st} in general

$$x(t) = e^{s_0 t} \rightarrow \boxed{H(s)} \rightarrow y(t) = H(s_0) e^{s_0 t}$$

$s_0 = \sigma_0 + j\omega_0$ Eigenfunction Property

→ Input signal has complex freq. s_0 , and so does output, but scaled by complex scalar $H(s_0)$

→ Laplace/Fourier transforms can describe how systems respond to input signals of different frequencies

Example: $H(s) = \frac{s}{s+100}$

$$x(t) = e^{j\pi t} + e^{j500\pi t} \rightarrow \boxed{H(s)} \rightarrow y(t)$$

$$\rightarrow H(j\pi) = \frac{j\pi}{j\pi + 100} = 0.0314 e^{j1.5394}$$

$$\rightarrow H(j500\pi) = \frac{j500\pi}{j500\pi + 100} = 0.998 e^{j0.0636}$$

$$\rightarrow y(t) = 0.0314 e^{j(\pi t + 1.5394)} + 0.998 e^{j(500\pi t + 0.0636)}$$

Response to Causal Sinusoids:

Eigenfunction property only works if input is on from $t = -\infty$ to $t = \infty$

- A good approximation sometimes, but not always
- Systems described by Laplace can also be analyzed in terms of how they respond to causal inputs — i.e., signals that turn on at $t = 0$

$$\rightarrow x(t) = e^{s_0 t} u(t)$$

— e.g., what happens in a circuit if I flip a switch to apply power?

- Causal sinusoidal signals have a "steady-state" response related to the eigenfunction response, as well as a "transient" response related to how the system responds to the input turning on

$$\text{Let } x(t) = e^{s_0 t} u(t) \xleftrightarrow{\mathcal{L}} X(s) = \frac{1}{s - s_0} \quad (s_0 \text{ is complex})$$

$$\text{Suppose system has rational } H(s) = \frac{P(s)}{Q(s)}$$

$$\rightarrow Y(s) = H(s)X(s) = \frac{P(s)}{\underbrace{(s - \lambda_1)(s - \lambda_2) \dots (s - \lambda_N)}_{\substack{\text{Called} \\ \text{"Characteristic Roots"} \\ \text{(Poles)}}} \underbrace{(s - s_0)}_{\substack{\text{Comes from} \\ \text{input signal}}}}$$

Performing PFE will give:

$$Y(s) = \sum_{i=1}^N \frac{K_i}{s - \lambda_i} + \frac{H(s_0)}{s - s_0}$$

$$\xleftrightarrow{\mathcal{L}^{-1}} y(t) = \underbrace{\sum_{i=1}^N K_i e^{\lambda_i t} u(t)}_{\text{"Transient" Response } y_{tr}(t)} + \underbrace{H(s_0) e^{s_0 t} u(t)}_{\text{"Steady-State" Response } y_{ss}(t)}$$

For stable systems, $\text{Re}\{\lambda_i\} < 0$

→ All transient terms decay to 0, leaving only $y_{ss}(t)$ as $t \rightarrow \infty$

Transient response terms are set by the poles of a system — we can tell a lot about transient response from pole/zero plot

Example: Basic step function, $s_0 = 0 \rightarrow x(t) = u(t) \leftrightarrow X(s) = \frac{1}{s}$

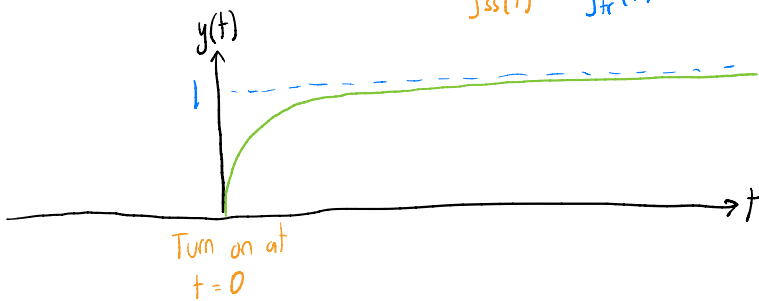
Let $H(s) = \frac{1}{s+1}$ (LPF) $\rightarrow Y(s) = H(s)X(s) = \frac{1}{(s+1)s} = \frac{A}{s+1} + \frac{B}{s}$

$\lambda_1 = -1$

$$A = (s+1)Y(s)|_{s=-1} = \frac{1}{-1} = -1$$

$$B = sY(s)|_{s=0} = \frac{1}{0+1} = 1$$

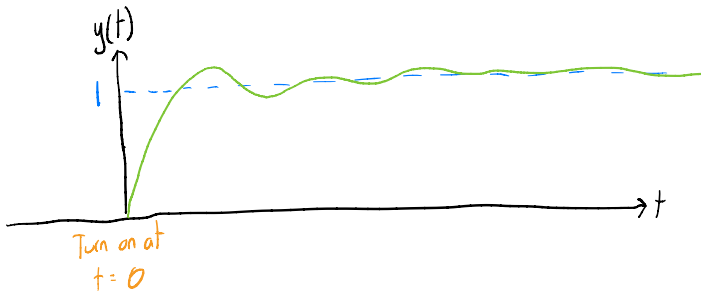
$$\rightarrow Y(s) = \frac{1}{s} - \frac{1}{s+1} \xleftrightarrow{\mathcal{L}^{-1}} y(t) = \underbrace{u(t)}_{y_{ss}(t)} - \underbrace{e^{-t}u(t)}_{y_{tr}(t)}$$



$\rightarrow$ approaches $u(t)$ as $t \rightarrow \infty$
 $\rightarrow y_{tr}(t)$ decays, and we are left with $y_{ss}(t)$

What about complex characteristic roots/poles? e.g., $H(s) = \frac{1}{(s+1+j\pi)(s+1-j\pi)}$

$\rightarrow$ We will observe sinusoidal transients related to the jw. component (e.g., $\omega_0 = \pi$ here)



$\rightarrow$ approaches $u(t)$ as $t \rightarrow \infty$
 $\rightarrow y_{tr}(t)$ overshoots/oscillates, but still decays to 0 and we are left with $y_{ss}(t)$

Transient analysis is important in circuit design and control systems, among others