System Characterization (Chapter 1.6)

This class is focused on analyzing signals and systems and their relationship

We focus on a class of systems called $\boxed{\text{linear systems}}$

- More specifically, we focus on <u>linear</u> <u>time-invariant</u> (LTI) systems (more on the TI later)

why?

- Many practical systems (e.g., RLC circuits, op amps, etc.) are linear or approx. linear
- Linear systems have nice properties → easy to analyze (compared to non-linear)

Notation:

$$\underbrace{y(t)}_{\text{Output}} = \underbrace{\mathcal{H}}_{\text{System operator}} \{ \underbrace{x(t)}_{\text{Input}} \}$$

General system model

Systems take in a <u>function</u> as input and outputs another <u>function</u>

Signals modeled as functions

Suppose $x(t) = a_1 x_1(t) + a_2 x_2(t)$

<u>If and only if</u> a system $\mathcal{H}$ is linear, then

$$y(t) = \mathcal{H}\{a_1 x_1(t) + a_2 x_2(t)\} = a_1 \mathcal{H}\{x_1(t)\} + a_2 \mathcal{H}\{x_2(t)\}$$

Linear system model

Components of Linearity

1. Scaling (or homogeneity)
2. Additivity

Scaling Property: amplitude scaling of input = identical scaling of output

$$H\{x(t)\} = y(t) \rightarrow H\{a\,x(t)\} = a\,y(t)$$

Ex. $y(t) = \sin(\pi t)\,x(t)$

Let $x_1(t) = 3x(t) \rightarrow y_1(t) = \sin(\pi t)\,x_1(t) = 3\sin(\pi t)\,x(t) = 3y(t)$ ✓

Ex. $y(t) = (x(t))^2$

Let $x_1(t) = 3x(t) \rightarrow y_1(t) = (x_1(t))^2 = (3x(t))^2 = 9(x(t))^2 \neq 3y(t)$ X

Additivity Property: Adding two input signals produces the same output as the sum of the original inputs separately

Let $H\{x_1(t)\} = y_1(t)$, $H\{x_2(t)\} = y_2(t) \rightarrow H\{x_1(t) + x_2(t)\} = y_1(t) + y_2(t)$

Ex. $y(t) = \sin(\pi t)\,x(t) \rightarrow y_1(t) = \sin(\pi t)\,x_1(t)$, $y_2(t) = \sin(\pi t)\,x_2(t)$

Let $x_3(t) = x_1(t) + x_2(t) \rightarrow y_3(t) = \sin(\pi t)\big(x_1(t) + x_2(t)\big)$

$= \sin(\pi t)\,x_1(t) + \sin(\pi t)\,x_2(t) = y_1(t) + y_2(t)$ ✓

Ex. $y(t) = (x(t))^2 \rightarrow y_1(t) = (x_1(t))^2$, $y_2(t) = (x_2(t))^2$

Let $x_3(t) = x_1(t) + x_2(t) \rightarrow y_3(t) = (x_1(t) + x_2(t))^2 = (x_1(t))^2 + 2x_1(t)x_2(t) + (x_2(t))^2$

$\neq y_1(t) + y_2(t)$ X

- Concepts sometimes referred to as superposition
- Linear systems must satisfy both properties

Proving Linearity:

- Need to show that the above 2 properties hold
- Can do this all at once as well
  - Show that if $H\{x_1(t)\} = y_1(t)$, $H\{x_2(t)\} = y_2(t)$,
    then $\boxed{H\{a_1x_1(t) + a_2x_2(t)\} = a_1y_1(t) + a_2y_2(t)}$
  - More generally, $H\left\{\sum_{k=1}^{K} a_k x_k(t)\right\} = \sum_{k=1}^{K} a_k y_k(t)$

Need to prove this

Can be shown in reverse also if it is easier to do

Steps:

1. Define $x_1(t) / x_1[n]$ and $x_2(t) / x_2[n]$

   $\rightarrow y_1(t) = H\{x_1(t)\}$ and $y_2(t) = H\{x_2(t)\}$

2. Define $x_3(t) / x_3[n]$ as scaled sum

   $\rightarrow x_3(t) = a_1 x_1(t) + a_2 x_2(t)$

   $\rightarrow y_3(t) = H\{x_3(t)\}$ → Scaled & summed at sytem <u>input</u>

3. Show whether

   $y_3(t) = a_1 y_1(t) + a_2 y_2(t)$ → Scaled & summed at system <u>output</u>

If equivalent → Linear ✓

If not equivalent → Not linear X

Example: $y[n] = m x[n] + b$

1. Define $x_1[n]$ & $x_2[n]$ $\longrightarrow$ $y_1[n] = m x_1[n] + b$

$y_2[n] = m x_2[n] + b$

2. Define $x_3[n] = a_1 x_1[n] + a_2 x_2[n]$

$\longrightarrow y_3[n] = m x_3[n] + b$

3. Does $y_3[n] = a_1 y_1[n] + a_2 y_2[n]$?

$a_1 y_1[n] + a_2 y_2[n] = a_1 m x_1[n] + a_1 b + a_2 m x_2[n] + a_2 b$

Not equal

$y_3[n] = m x_3[n] + b = m\left(a_1 x_1[n] + a_2 x_2[n]\right) + b$

$= a_1 m x_1[n] + a_2 m x_2[n] + b$

$\longrightarrow$ Not linear

Example: $v_i(t) = v_o(t) + RC \dfrac{dv_o(t)}{dt}$ $\qquad v_o(t) = \mathcal{H}\{v_i(t)\}$

1. Define $y_1(t) = \mathcal{H}\{x_1(t)\}$ and $y_2(t) = \mathcal{H}\{x_2(t)\}$

$\longrightarrow x_1(t) = y_1(t) + RC \dfrac{dy_1(t)}{dt}$, $\quad x_2(t) = y_2(t) + RC \dfrac{dy_2(t)}{dt}$

2. Define $x_3(t) = a_1 x_1(t) + a_2 x_2(t)$

$\longrightarrow x_3(t) = y_3(t) + RC \dfrac{dy_3(t)}{dt}$

3. Does $y_3(t) = a_1 y_1(t) + a_2 y_2(t)$?

This is a little harder – but we can prove the reverse:

i.e., if $y_3(t) = a_1 y_1(t) + a_2 y_2(t)$, does $x_3(t) = a_1 x_1(t) + a_2 x_2(t)$?

Let $\underline{y(t) = a_1 y_1(t) + a_2 y_2(t)}$

Then $x(t) = y(t) + RC \frac{dy(t)}{dt} = a_1 y_1(t) + a_2 y_2(t) + \underbrace{RC \frac{d}{dt}(a_1 y_1(t) + a_2 y_2(t))}_{\text{separate}}$

$$= a_1 y_1(t) + a_2 y_2(t) + a_1 RC \frac{dy_1(t)}{dt} + a_2 RC \frac{dy_2(t)}{dt}$$

$$= a_1 \underbrace{\left(y_1(t) + RC \frac{dy_1(t)}{dt}\right)}_{x_1(t)} + a_2 \underbrace{\left(y_2(t) + RC \frac{dy_2(t)}{dt}\right)}_{x_2(t)}$$

$$= a_1 x_1(t) + a_2 x_2(t) = \underline{x_3(t)}$$

$\rightarrow$ A superposition at the output implies the same superposition at the input

$\rightarrow$ Linear

Note: ANY <u>differential equation</u> w/ constant coefficients in a linear form defines a linear system (e.g., any RLC circuit)

Example: $\frac{dx(t)}{dt} + 3x(t) = \frac{d^2y(t)}{dt^2} + 4y(t) \longrightarrow$ linear

Example: $x(t)\frac{dx(t)}{dt} = \frac{d^2y(t)}{dt} \longrightarrow$ Not linear

There is an analogous equation in DT called a <u>difference equation</u> which is also always linear $\rightarrow$ more on this later in the semester

Time - Invariance

- LTI systems are linear and time-invariant

A system is time-invariant if a delay in the input by t_0 seconds or n_0 samples produces the same output signal, just delayed by t_0 seconds or n_0 samples

- Time shift at input produces identical time shift at output
- I/O characteristics do not change w/ time

Mathematically: $\mathcal{H}\{x(t)\} = y(t) \rightarrow \mathcal{H}\{x(t-t_0)\} = y(t-t_0)$

for all $x(t)$ and t_0

Proving Time - Invariance

1. Define $x_1(t)$ w/ $y_1(t) = \mathcal{H}\{x_1(t)\}$
2. Define $x_2(t) = x_1(t-t_0) \rightarrow y_2(t) = \mathcal{H}\{x_2(t)\}$
3. Show whether $y_1(t-t_0) = y_2(t)$

If equivalent $\rightarrow$ Time-invariant ✓

If not $\rightarrow$ Time-varying ✗

Example: $y(t) = x(t^2)$

1. Define $x_1(t) \rightarrow y_1(t) = x_1(t^2)$
2. Define $x_2(t) = x_1(t-t_0) \rightarrow y_2(t) = \mathcal{H}\{x_2(t)\}$
3. Does $y_1(t-t_0) = y_2(t)$?

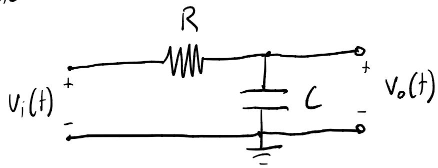
$$y_2(t) = x_2(t^2) = x_1(t^2 - t_0)$$

$$y_1(t-t_0) = x_1((t-t_0)^2) = x_1(t^2 - 2tt_0 + t_0^2)$$

Not equal $\rightarrow$ Time-varying

Why LTI?

Consider the basic RC circuit:



$$\rightarrow v_o(t) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} v_i(t) = \underbrace{\frac{1}{1 + j\omega RC}}_{\text{Defined for specific values of } \omega} v_i(t) = \mathcal{H}\{v_i(t)\}$$

Suppose $R = 500\ \Omega$ and $C = 5\ nF$, $v_i(t) = \underbrace{e^{j\omega t}}_{\text{Only one } \omega \text{ value}}$

- At $\omega = 0$ rad/s $\rightarrow v_i(t) = 1$ (DC) $\rightarrow v_o(t) = \frac{1}{1 + j(0)RC}(1) = 1 = v_i(t) \rightarrow$ Unchanged
- At $\omega = 400{,}000$ rad/s $\rightarrow v_i(t) = e^{j400{,}000t} \rightarrow v_o(t) = \frac{1}{1 + j400{,}000RC} e^{j400{,}000t}$

$$= 0.707e^{j400{,}000t} e^{-j\pi/4} = \underbrace{0.707e^{-j\pi/4}}_{\text{Original input scaled by half power \& } 45^\circ \text{ phase shift}} v_i(t)$$

- At $\omega = 40$ Mrad/s $\rightarrow v_i(t) = e^{j(40 \cdot 10^6)t} \rightarrow v_o(t) = \frac{1}{1 + j(40 \cdot 10^6)RC} e^{j(40 \cdot 10^6)t}$

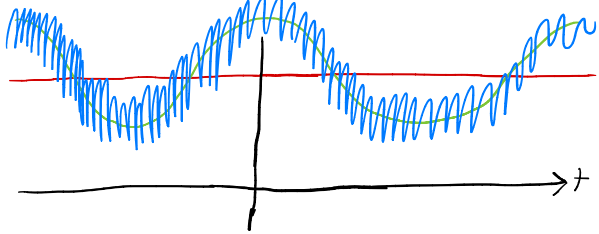
$$\simeq 0.01e^{j(40 \cdot 10^6)t} e^{-j\frac{\pi}{2}} = \underbrace{0.01e^{-j\pi/2}}_{\text{Original input scaled by } 10^{-4} \text{ in power \& } 90^\circ \text{ phase shift}} v_i(t)$$

Notice:

- Each input frequency produces an output sinusoid at the same frequency
- Each frequency is scaled with a different amplitude and phase shift (a complex scale factor) defined as a function of $\omega \rightarrow$ Call this $H(\omega)$

What if input is multiple frequencies?

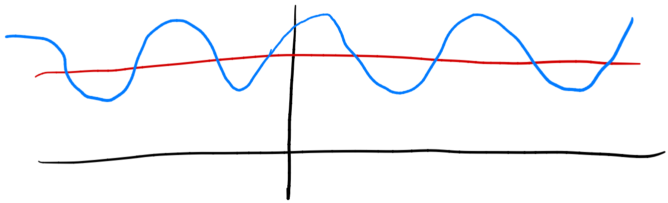
e.g., $v_i(t) = 1 + e^{j400{,}000t} + e^{j(40\cdot 10^6)t}$



$Re\{v_i(t)\}$

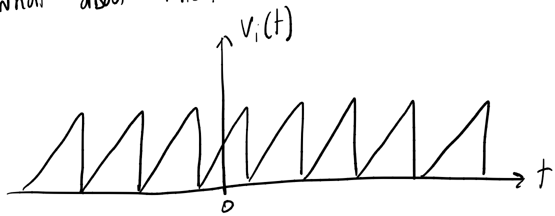
Remember linearity! Scale & phase shift each sinusoid separately, then add back together:

$$v_o(t) = 1 + 0.707e^{-j\pi/4}e^{j400{,}000t} + 0.01e^{-j\pi/2}e^{j(40\cdot 10^6)t}$$



$$\text{So, } v_o(t) = \underbrace{\frac{1}{1+j\omega RC}}_{\text{Frequency-dependent scale factor } H(\omega)} v_i(t)$$

What about this?



???

- Complicated signal, no obvious discrete sinusoids...
- It turns out we can apply a transform to $v_i(t)$ to get

$$v_i(t) = a_1e^{j\omega_1 t} + a_2e^{j\omega_2 t} + a_3e^{j\omega_3 t} + \ldots$$

So at the output we can compute:

$$V_o(t) = \underline{a_1 H(\omega_1) e^{j\omega_1 t}} + \underline{a_2 H(\omega_2) e^{j\omega_2 t}} + \underline{a_3 H(\omega_3) e^{j\omega_3 t}} + \ldots$$

Each frequency comes out of the LTI system as the same frequency, just scaled and phase shifted by $H(\omega)$ independently of all other frequencies

Like eigenvectors in matrix algebra $Ax = \lambda x$, but now $H\{e^{j\omega_i t}\} = H(\omega_i) e^{j\omega_i t}$

This fundamental relationship between LTI systems and sinusoids makes LTI systems easier to work with and sinusoids convenient for constructing signals.

The transform we applied above is called the continuous-time Fourier Series.

In this class we will learn the following transforms:

- CT and DT periodic signals: Fourier Series
- CT aperiodic signals: Fourier (Ω or F) and Laplace (s) Transforms
- DT aperiodic signals: Discrete-Time Fourier (ω or f) and z- (z) Transforms

Causality

A system is causal if output at $t = t_0$ only depends on input for $t \le t_0$

- System output doesn't require prediction of future values
- A requirement for real-time systems

Examples

$y(t) = -x(t-2) + x(t+3) \rightarrow$ Try $t=0 \rightarrow$ requires $x(3) \rightarrow$ Non-causal

$y[n] = \sum_{k=0}^{M} a_k x[n-k] \rightarrow$ Try $n=0 \rightarrow$ requires $x[0], \ldots, x[M] \rightarrow$ Causal

$y[n] = \sum_{k=-1}^{M} a_k x[n-k] \rightarrow$ Try $n=0 \rightarrow$ requires $x[1] \rightarrow$ Non-causal

Memoryless vs. With Memory

- Memoryless systems have outputs at each value of t or n that only depend on values of the input at that time
  - Resistor: $v(t) = Ri(t) \rightarrow$ Memoryless
  - Capacitor: $v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \rightarrow$ has (infinite) memory

Invertible Systems

- In invertible systems, each unique input produces a unique output
- If a system is invertible, there exists an "inverse" system that produces the original input signal from the output signal

  i.e., if $\mathcal{H}\{x(t)\} = y(t)$, then $\mathcal{H}^{-1}\{y(t)\} = x(t)$
- Example: $y(t) = (x(t))^2$ is not invertible

Bounded-Input, Bounded-Output (BIBO) Stable

- If the input to a BIBO stable system is bounded/finite, the output will also be bounded/finite (no divergence)
- Unstable example: $y[n] = \sum_{k=0}^{\infty} x[n-k]$ (Try $x[n] = u[n]$)

LTI Example:

Suppose that for some system H, when the input $x(t) = \delta(t)$, the output $y(t) = e^{-t}u(t)$.

Now, when the input is $x_1(t) = 3\delta(t) - 2\delta(t-2)$, find the output $y_1(t)$.

Linearity: $Ax(t) \xrightarrow{H} Ay(t)$ and $x_1(t) + x_2(t) \xrightarrow{H} y_1(t) + y_2(t)$

We can find the response to each $\delta(t)$ and add together results!

Also, the scaling factors follow through an LTI system.

$\rightarrow 3\delta(t) \xrightarrow{H} 3e^{-t}u(t)$

What about the time-shift in $-2\delta(t-2)$?

Time-Invariance: $x(t-t_0) \xrightarrow{H} y(t-t_0)$

We can use the same $y(t)$, just shifted!

$\rightarrow -2\delta(t-2) \xrightarrow{H} -2e^{-(t-2)}u(t-2)$

$\rightarrow y_1(t) = 3e^{-t}u(t) - 2e^{-(t-2)}u(t-2)$

Another way: $x(t) = \delta(t) \xrightarrow{H} y(t) = e^{-t}u(t) \longrightarrow x_1(t) = 3\delta(t) - 2\delta(t-2) = 3x(t) - 2x(t-2)$

$\xrightarrow{H} y_1(t) = 3y(t) - 2y(t-2) = 3e^{-t}u(t) - 2e^{-(t-2)}u(t-2)$

Another Example:

$x(t) = 4\delta(t-1) \rightarrow \boxed{H} \rightarrow y(t) = 8\cos\left(\frac{\pi}{2}(t-2)\right)u(t-2)$

Find $y_1(t)$ when $x_1(t) = 3\delta(t+1)$.

First: work backward to find $y_2(t)$ when $x_2(t) = \delta(t)$

$x(t) = 4\delta(t-1) \xrightarrow{H} y(t) = 8\cos\left(\frac{\pi}{2}(t-2)\right)u(t-2) \longrightarrow x_2(t) = \delta(t) \xrightarrow{H} y_2(t) = \frac{1}{4}y(t+1)$

$= 2\cos\left(\frac{\pi}{2}(t-1)\right)u(t-1) = y_2(t)$

$\rightarrow$ So: $x_1(t) = 3\delta(t+1) \xrightarrow{H} y_1(t) = 3y_2(t+1) = 6\cos\left(\frac{\pi}{2}t\right)u(t)$