Filters

High-level discussion - Not design, just concepts

How to design real filters?

$\rightarrow$ Circuits II, RF & Microwave Filter Design, DSP

LTI system

$$x(t) \longrightarrow \boxed{\begin{matrix} h(t) \\ H(F) \end{matrix}} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t) \longrightarrow Y(F) = X(F)\underbrace{H(F)}_{\text{System Freq. Response}}$$

(Assume stable!!! FT must exist)

$H(F)$ can change amplitude as well as relative phases of frequency components

$\rightarrow$ change in phase <u>distorts</u> the signal

Suppose a system freq. response has const. amplitude and linear phase

$$H(F) = Ge^{-j2\pi F t_0}$$

So, for some input $x(t) \xrightarrow{\mathcal{F}} X(F)$, $\quad Y(F) = X(F)H(F) = X(F)Ge^{-j2\pi F t_0}$

$$\rightarrow y(t) = G\mathcal{F}^{-1}\{X(F)e^{-j2\pi F t_0}\} = \boxed{Gx(t-t_0)}$$

$\rightarrow$ This system is "distortionless"

- $x(t)$ is amplified and delayed but its shape is unchanged
- "Linear Phase" systems delay all freq. components the same

Group Delay: Frequency-dependent delay of freq-components through a system

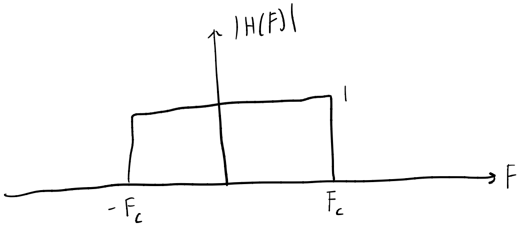
$$t_g(F) \triangleq -\frac{1}{2\pi}\frac{d}{dF}\angle H(F) \quad \left(\text{or, } t_g(\omega) = -\frac{d}{d\omega}\angle H(\omega)\right)$$

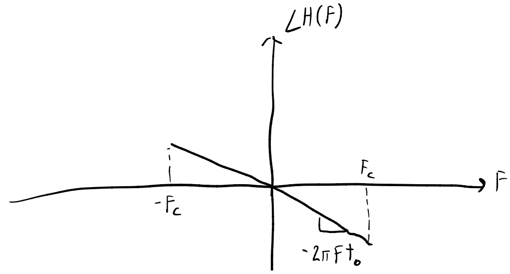
Ex. $H(F) = Ge^{-j2\pi F t_0} \rightarrow \angle H(F) = -2\pi F t_0 \rightarrow t_g(F) = -\frac{1}{2\pi}\frac{d}{dF}(-2\pi F t_0) = \underrightarrow{t_0}$: constant

$\therefore$ We want linear phase (or equivalently, flat/constant group delay) and flat/constant amplitude to pass signals undistorted

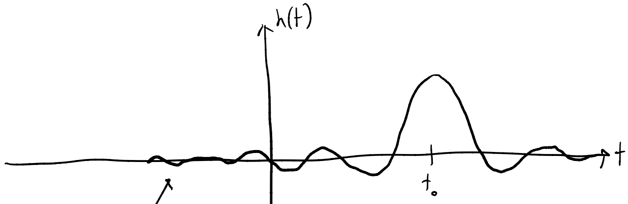
Ideal Filters:

Ideal: Flat in passband, zero in stopband, and linear phase in passband





$\rightarrow H(F) = \text{rect}\left(\frac{F}{2F_c}\right)e^{-j2\pi F t_0}$ $\rightarrow h(t) = 2F_c\,\text{sinc}(2F_c(t - t_0))$



$h(t) = 0$ when $t<0$

$\rightarrow$ non-causal!

Can't be implemented in real-time $\rightarrow$ not realizable

Paley-Wiener Criterion

A system w/ a causal impulse response must have $|H(\omega)|$ that satisfies

$$\int_{-\infty}^{\infty} \frac{|\ln|H(\omega)||}{1+\omega^2}\,d\omega < \infty$$

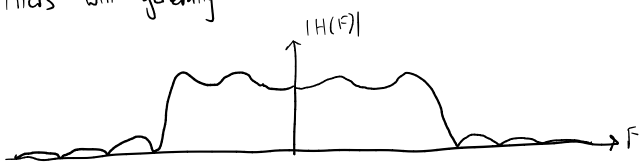
Derivation? Beats me.

Note: When $|H(\omega)| = 0$, $|\ln|H(\omega)|| = +\infty$

$\rightarrow |H(\omega)|$ can equal 0 at single frequencies b/c height of $|\ln|H(\omega)|| = \infty$ but width $= 0$

BUT $|H(\omega)|$ cannot equal 0 for any continuous band or PW criterion not satisfied

Result: Practical filters will generally have some kind of ripple



Making practical filters?

- One approach: Cut off tails of ideal $h(t)$ w/ a "window" function $w(t)$

$$\hat{h}(t) = h(t)w(t)$$

$\hat{h}(t)$ approximates $h(t)$, $\hat{H}(F)$ approximates $H(F)$

Example: Project 2 LPF was the ideal sinc impulse response but truncated

→ $w(t)$ is a rectangular window

→ Truncation of $h(t)$ is why there is ripple and roll-off instead of ideal rect

→ Different windows than rect exist to give better performance

- Hanning, Hamming, Blackman, etc. → Take DSP to learn more

Digital Filters

Same concepts in causality/realizability as CT filters

Two filter types: Finite Impulse Response (FIR) and Infinite Impulse Response (IIR)

Both filters implemented by LCCDE's

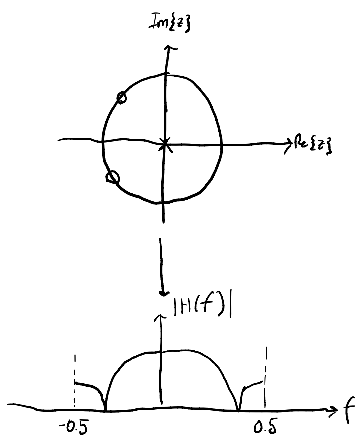
FIR: $$y[n] = b_0 x[n] + b_1 x[n-1] + \dots + b_M x[n-M] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\rightarrow h[n] = b_0 \delta[n] + b_1 \delta[n-1] + \dots + b_M x[n-M] = \sum_{k=0}^{M} b_k \delta[n-k]$$

$$\rightarrow H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

Advantages:

- Easy to make linear-phase
- Only has poles at $z=0$ → guaranteed to be stable



Disadvantages:

- Requires many more delays and adds than an IIR filter to get the same performance

IIR: $a_0 y[n] + \ldots + a_N y[n-N] = b_0 x[n] + \ldots + b_M x[n-M]$

$$\rightarrow \sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

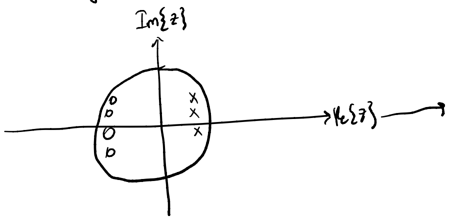
$$\rightarrow H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \quad \begin{matrix} \leftarrow \text{zeros} \\ \leftarrow \text{poles} \end{matrix}$$

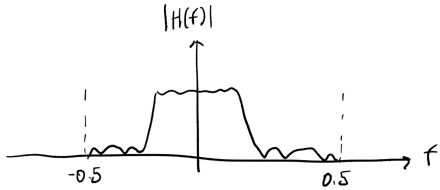
Advantages:

- Can better performance / steeper cutoffs for same filter order

Disadvantages:

- Need to design such that poles all inside unit circle (stability)
- Can't get perfect linear phase





By the way, CT filters have similar interpretations in terms of the pole-zero plot of the Laplace Transform $H(s)$

Show window-designer in Matlab