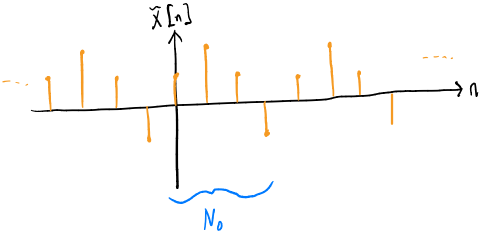
Discrete-Time Fourier Transform

Consider DT periodic signal $\tilde{x}[n]$



Can represent $\tilde{x}[n]$ in frequency domain using DT Fourier series

→ N_0 coefficients a_k

→ $a_0, a_1, \ldots, a_{N_0-1}$

$$\tilde{x}[n] = \sum_{k=\langle N_0 \rangle} a_k e^{j 2\pi \underbrace{k f_0}_{=f} n} = \sum_{k=\langle N_0 \rangle} a_k e^{j2\pi \underbrace{\left(\frac{k}{N_0}\right)}_{=f} n}$$

$$a_k = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} \tilde{x}[n] e^{-j2\pi \underbrace{\left(\frac{k}{N_0}\right)}_{=f} n}$$

Like CT FS, each integer k corresponds to a frequency

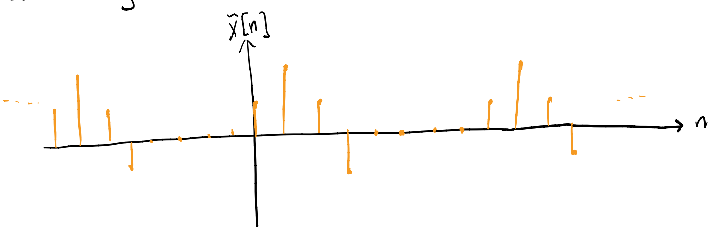
→ $f = k f_0 = \frac{k}{N_0}$

$\tilde{x}[n]$ has a certain "amount" of each frequency $f = kf_0 = \frac{k}{N_0}$ for $k = 0, 1, \ldots, N_0 - 1$

→ $f = 0, \frac{1}{N_0}, \frac{2}{N_0}, \ldots, \frac{N_0 - 1}{N_0}$

→ Each freq. component has a mag/phase set by a_k

Let's try "padding" the periodic signal with zeros

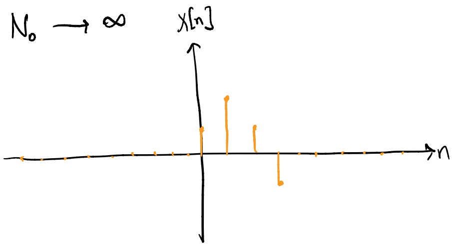


→ Periodic, but increased N_0

→ $f = \frac{k}{N_0}$ gets more closely spaced

Last frequency $f = \frac{N_0 - 1}{N_0} \to 1$

Let $N_0 \to \infty$



$\tilde{x}[n]$ above is periodic

→ Signal no longer periodic → $x[n]$

→ Freq. spacing $k/N_0 \to 0$

→ f becomes a continuous variable $0 \le f < 1$

So w/ $N_0 = \infty$, we have $a_k = \frac{1}{N_0} \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi(\frac{k}{N_0})n}$

Define: $X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n}$

Then DT FS coefficients a_k given by $a_k = \frac{1}{N_0} X(f) \Big|_{f = \frac{k}{N_0} = kf_0} = f_0 X(f) \Big|_{f = kf_0}$

Plug this into reconstruction of periodic $\tilde{x}[n]$:

$$\tilde{x}[n] = \sum_{k=\langle N_0 \rangle} a_k e^{j2\pi k f_0 n} = \sum_{k=\langle N_0 \rangle} X(f = kf_0) e^{j2\pi k f_0 n} \cdot f_0$$

As $N_0 \to \infty$, $f_0 \to 0$, final frequency $f = \frac{N_0 - 1}{N_0} \to 1$, and summation $\to$ integral

$$\to \underset{\text{Periodic}}{\tilde{x}[n]} \to \underset{\text{Aperiodic}}{x[n]} = \int_0^1 X(f) e^{j2\pi f n} \, df$$

$\to$ Integral for reconstruction of aperiodic signal from continuous spectrum of freq. components!

Discrete-Time Fourier Transform (DTFT) Pair

$$x[n] = \int_{\langle 1 \rangle} X(f) e^{j2\pi f n} \, df = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\omega) e^{j\omega n} \, d\omega$$

$$X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n} \quad \text{or} \quad X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

From DT FS: a_k is periodic because frequencies are not unique

$\to$ Same concept in DTFT!

$X(f)$ is periodic w/ period of $1 \to 0 \le f < 1$ OR $-\frac{1}{2} \le f < \frac{1}{2}$

$X(\omega)$ is periodic w/ period of $2\pi \to 0 \le \omega < 2\pi$ OR $-\pi \le \omega < \pi$

Some DTFT Properties

1.) Signal energy

Aperiodic signal $x[n] \longrightarrow$ energy signal

Parseval's Theorem (again):

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \int_{\langle 1 \rangle} |X(f)|^2 df = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} |X(\omega)|^2 d\omega$$

2.) Symmetry: Same as other Fourier representations!

- $x[n]$ real-valued $\longrightarrow$ conjugate symmetry, $X(f) = X^*(-f)$
- $x[n]$ real & even $\longrightarrow$ $X(f)$ real-valued & even, $X(f) = X(-f)$
- $x[n]$ real & odd $\longrightarrow$ $X(f)$ imaginary-valued & odd, $X(f) = -X(-f)$

3.) Discreteness / Periodicity Duality

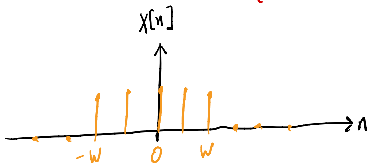
Signals that are discrete in one domain are periodic in the other

	Time-Domain	Frequency Domain
CT Fourier Transform:	Continuous, Aperiodic	Aperiodic, Continuous
CT Fourier Series:	Continuous, Periodic	Aperiodic, Discrete
DT Fourier Transform:	Discrete, Aperiodic	Periodic, Continuous
DT Fourier Series:	Discrete, Periodic	Periodic, Discrete

Example:

$$x[n] = \begin{cases} 1 & |n| \le W \\ 0 & \text{else} \end{cases}$$

(Sort of like a DT rect function)



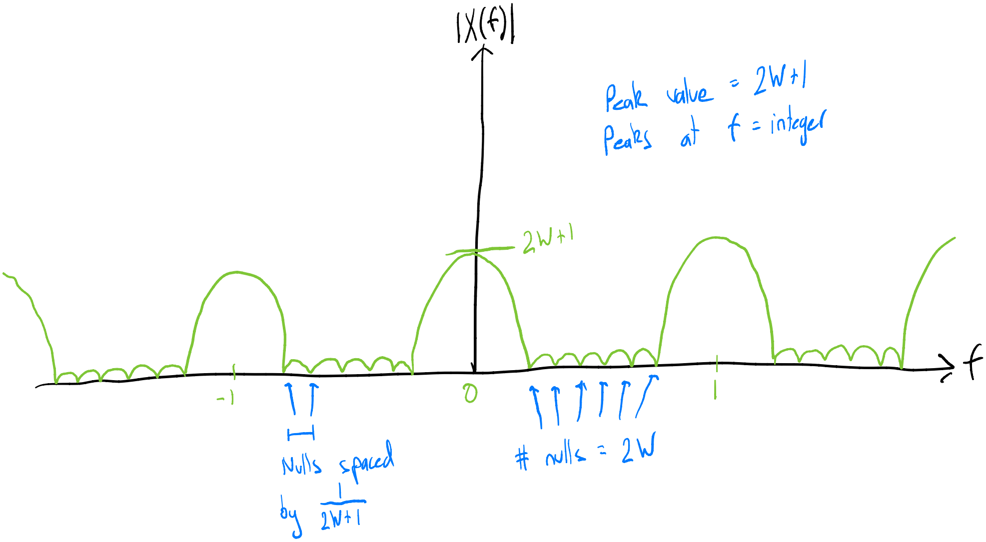
$$X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n} = \sum_{n=-W}^{W} (1) e^{-j2\pi f n} = \sum_{n=-W}^{W} (\underbrace{e^{-j2\pi f}}_{=a})^n = \frac{a^{-W} - a^{W+1}}{1 - a}$$

$$= \frac{e^{j2\pi f W} - e^{-j2\pi f(W+1)}}{1 - e^{-j2\pi f}} = \frac{e^{-j\pi f} e^{j2\pi f(W+\frac{1}{2})} - e^{-j\pi f} e^{-j2\pi f(W+\frac{1}{2})}}{e^{-j\pi f}(e^{j\pi f} - e^{-j\pi f})}$$

$$= \frac{2j \sin(2\pi f(W+\frac{1}{2}))}{2j \sin(\pi f)} = \boxed{\frac{\sin(\pi f(2W+1))}{\sin(\pi f)}}$$

→ Result: Sort of like a sinc function (rect/sinc duality)
But this one is periodic!

→ Numerator & denominator go to zero for integer f regardless of W!



Example: $x[n] = \delta[n] \longrightarrow X(f) = \sum_{n=0} (1) e^{-j2\pi f n} = \boxed{1}$

$$x[n] = \delta[n-n_0] \longrightarrow X(f) = \sum_{n=n_0} (1) e^{-j2\pi f n} = \boxed{e^{-j2\pi f n_0}}$$

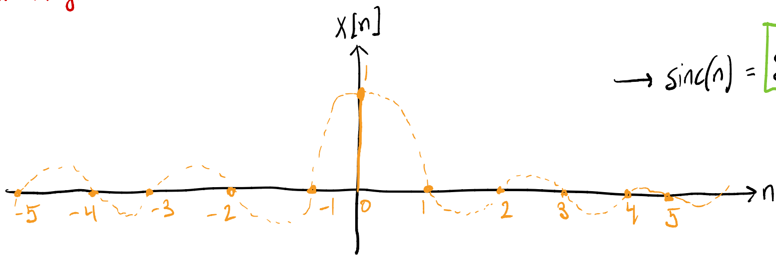
$$\boxed{\text{Property: } x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(f) \longrightarrow x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} X(f) e^{-j2\pi f n_0}}$$

Inverse DTFT Example: $X(f) = 1$

$$\rightarrow x[n] = \int_{\langle 1 \rangle} X(f) e^{j2\pi f n} df = \int_{-1/2}^{1/2} e^{j2\pi f n} df = \left. \frac{e^{j2\pi f n}}{j2\pi n} \right|_{-1/2}^{1/2}$$

$$= \frac{e^{j\pi n} - e^{-j\pi n}}{j2\pi n} = \frac{j2\sin(\pi n)}{j2\pi n} = \frac{\sin(\pi n)}{\pi n} = \boxed{\text{sinc}(n)}$$

$\rightarrow$ For integer n, $\text{sinc}(n) = 1$ at $n=0$ and $\text{sinc}(n) = 0$ for other n

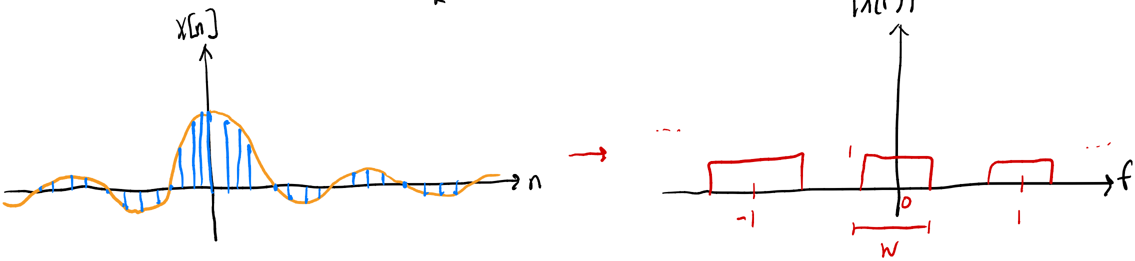


$$\longrightarrow \text{sinc}(n) = \boxed{\delta[n] = x[n]}$$

Other important DTFT pairs (on your table!):

$$x[n] = 1 \xleftrightarrow{\mathcal{F}} X(f) = \sum_{k=-\infty}^{\infty} \delta(f-k)$$

$$x[n] = W\operatorname{sinc}(Wn) \xleftrightarrow{\mathcal{F}} X(f) = \sum_{k=-\infty}^{\infty} \operatorname{rect}\left(\frac{f-k}{W}\right)$$



$$x[n] = \cos(2\pi f_0 n) \xleftrightarrow{\mathcal{F}} X(f) = \sum_{k=-\infty}^{\infty} \frac{1}{2}\left(\delta(f-f_0-k) + \delta(f+f_0-k)\right)$$

$$x[n] = a^n u[n] \quad (|a|<1) \xleftrightarrow{\mathcal{F}} X(f) = \frac{1}{1-ae^{-j2\pi f}}$$

Other Important Properties (on table):

Frequency Shifting: $x[n]e^{j2\pi f_0 n} \xleftrightarrow{\mathcal{F}} X(f-f_0)$

Convolution: $x[n] * h[n] \xleftrightarrow{\mathcal{F}} X(f)H(f)$ → Just like CT! $h[n]$ impulse response

→ $H(f)$ frequency response for DT systems

Multiplication: $x[n]y[n] \xleftrightarrow{\mathcal{F}} X(f) * Y(f)$ (periodic convolution... we'll probably try to avoid this property)

☆ Pairs and properties table for DTFT

→ Use just like CT FT to evaluate complex DTFTs without direct evaluation

→ Note: DTFT of short, finite sequences is easy to compute directly from the summation definition

Example: $x[n] = 2(0.9)^n \cos\left(\frac{\pi}{4}n\right) u[n] \rightarrow$ Find DTFT $X(f)$

→ Like CT, break into chunks and use pairs & properties

→ For these, unlike CT, we want to avoid freq. convolution, but con
apply frequency shifting instead

$\rightarrow \cos\left(\frac{\pi}{4}n\right) = \frac{1}{2}e^{j\frac{\pi}{4}n} + \frac{1}{2}e^{-j\frac{\pi}{4}n}$

$\rightarrow x[n] = (0.9)^n e^{-j\frac{\pi}{4}n} + (0.9)^n e^{j\frac{\pi}{4}n}$

Pair: $a^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{1}{1 - ae^{-j2\pi f}}$

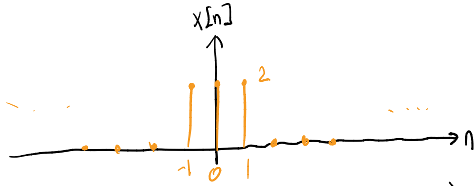
$\rightarrow (0.9)^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{1}{1 - 0.9e^{-j2\pi f}}$

Frequency-Shifting Property: $x[n]e^{j2\pi f_0 n} \overset{\mathcal{F}}{\longleftrightarrow} X(f - f_0)$

For $e^{\pm j\frac{\pi}{4}n} \rightarrow f_0 = \pm\frac{1}{8}$

$$\rightarrow \boxed{X(f) = \frac{1}{1 - 0.9e^{j2\pi\left(f - \frac{1}{8}\right)}} + \frac{1}{1 - 0.9e^{j2\pi\left(f + \frac{1}{8}\right)}}}$$

Another Example:



a.) Find DTFT of $\tilde{x}[n] = 2x[n]\cos\left(2\pi\left(\frac{1}{3}\right)n\right)$

First: $x[n] = \begin{cases} 2 & |n| \le 1 \\ 0 & \text{else} \end{cases} \overset{\mathcal{F}}{\longleftrightarrow} X(f) = \dfrac{\sin(\pi f(2W+1))}{\sin(\pi f)} = 2\dfrac{\sin(3\pi f)}{\sin(\pi f)}$

Euler's: $\tilde{x}[n] = 2x[n]\left(\frac{1}{2}e^{j2\pi(\frac{1}{3})n} + \frac{1}{2}e^{-j2\pi(\frac{1}{3})n}\right) = x[n]e^{j2\pi(\frac{1}{3})n} + x[n]e^{-j2\pi(\frac{1}{3})n}$

$\rightarrow$ Frequency shift by $f_0 = \pm\frac{1}{3}$

$\rightarrow \tilde{X}(f) = X\left(f-\frac{1}{3}\right) + X\left(f+\frac{1}{3}\right) = \boxed{2\dfrac{\sin\left(3\pi\left(f-\frac{1}{3}\right)\right)}{\sin\left(\pi\left(f-\frac{1}{3}\right)\right)} + 2\dfrac{\sin\left(3\pi\left(f+\frac{1}{3}\right)\right)}{\sin\left(\pi\left(f+\frac{1}{3}\right)\right)}}$

b.) Let $h[n] = \delta[n-10]$ Find $H(f)$.

Pair: $\delta[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j2\pi f n_0}$

$\rightarrow \boxed{H(f) = e^{-j20\pi f}}$

c.) Let $y[n] = \tilde{x}[n] * h[n]$. Find $Y(f)$ and take the Inverse DTFT to find $y[n]$.

Convolution: $Y(f) = \tilde{X}(f)H(f) = \boxed{2\left(\dfrac{\sin\left(3\pi\left(f-\frac{1}{3}\right)\right)}{\sin\left(\pi\left(f-\frac{1}{3}\right)\right)} + \dfrac{\sin\left(3\pi\left(f+\frac{1}{3}\right)\right)}{\sin\left(\pi\left(f+\frac{1}{3}\right)\right)}\right)e^{-j20\pi f}}$

Phase shift in freq.
$\rightarrow$ Time shift

$y[n]$? $\rightarrow 2\dfrac{\sin(3\pi(f-f_0))}{\sin(\pi(f-f_0))} \overset{\mathcal{F}^{-1}}{\longleftrightarrow} x[n]e^{j2\pi f_0 n}$

$2\dfrac{\sin(3\pi(f-f_0))}{\sin(\pi(f-f_0))}e^{-j2\pi f n_0} \overset{\mathcal{F}^{-1}}{\longleftrightarrow} x[n-n_0]e^{j2\pi f(n-n_0)}$

$\rightarrow y[n] = x[n-10]e^{j2\pi(\frac{1}{3})(n-10)} + x[n-10]e^{-j2\pi(\frac{1}{3})(n-10)} = \boxed{2x[n-10]\cos\left(2\pi\left(\frac{1}{3}\right)(n-10)\right)}$

$= \tilde{x}[n-10] = \tilde{x}[n] * \delta[n-10]$