DT Complex Exponentials

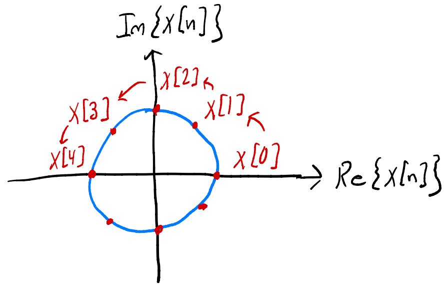
Recall DT complex sinusoids:

$$x[n] = e^{j\omega_o n} \qquad \omega_o \text{ is in rad/sample, not rad/s}$$

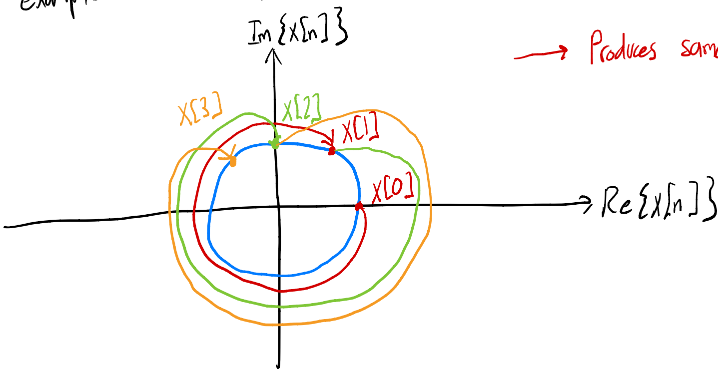
$$= e^{j2\pi f n} \qquad f_o \text{ is in cycles/sample}$$

Each sample gives an extra rotation of ω_o rad around the unit circle:

Ex. $\omega_o = \pi/4 \longrightarrow x[n] = e^{j\frac{\pi}{4}n}$



Another example: $\omega_o = -\frac{7\pi}{4} \longrightarrow x[n] = e^{-j\frac{7\pi}{4}n}$



→ Produces same sequence as when $\omega_o = \frac{\pi}{4}$

→ In DT, frequencies are not unique!

Let $\omega_1 = \omega_o + 2\pi \longrightarrow x[n] = e^{j\omega_1 n} = e^{j\omega_o n + j2\pi n} = e^{j\omega_o n}\underbrace{e^{j2\pi n}}_{=1} = e^{j\omega_o n}$

→ Adding/subtracting integer multiple of 2π from ω_o yields an identical DT frequency

Aliasing: No distiction between frequencies spaced by 2π, so usually assume it is lowest possible frequency – If it was actually a higher frequency, we will never know, and lose information!

Result: In DT, we usually restrict "meaningful" frequencies to

ω: $-\pi \leq \omega < \pi$ or $0 \leq \omega < 2\pi$

f: $-\frac{1}{2} \leq f < \frac{1}{2}$ or $0 \leq f < 1$

Other frequencies exist, but they ar redundant

Important concept: Normalized Frequency

- $x[n]$ usually sampled version of CT $x(t)$ w/ sampling freq. F_s
- Samples spaced by $T_s = 1/F_s$

→ We write sample index as integer n – essentially "normalizing" sampling period to 1

→ Must also normalize frequency to F_s

→ $\boxed{f = F/F_s}$

CT frequency – Upper-case

Sampling frequency

Normalized frequency
Lower-case

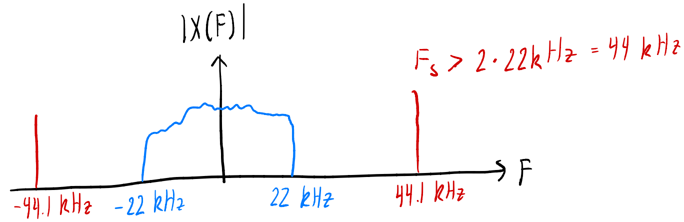
From above: $-\frac{1}{2} \leq f < \frac{1}{2}$ $\xrightarrow{f = F/F_s}$ $-\frac{1}{2} \leq \frac{F}{F_s} < \frac{1}{2}$

→ $\boxed{-\frac{F_s}{2} \leq F < \frac{F_s}{2}}$

→ To avoid aliasing, all frequency content in CT signal must fall in this range!

In other words, the sampling frequency F_s must be more than twice the highest frequency of CT signal to avoid aliasing → $\boxed{\text{Nyquist-Shannon Sampling Theorem}}$

Example: Audio



Discrete-Time Fourier Series

Same concept as CT Fourier series - Periodic $x[n]$ represented as sum of harmonics

Periodic DT: $x[n] = x[n+N_0]$ for positive integer N_0 & all n

Smallest possible $N_0 \triangleq$ Fundamental period

$f_0 = 1/N_0 \qquad \omega_0 = 2\pi/N_0$ Fundamental frequency

cycles/sample $\qquad$ rad/sample

$\left.\begin{matrix} k\omega_0 \\ kf_0 \end{matrix}\right\} \rightarrow$ For integer k: k^{th} harmonic frequency

Harmonic sinusoid: $\phi_k[n] = e^{j2\pi(kf_0)n} = e^{j2\pi\left(\frac{k}{N_0}\right)n}$

Note: DT exponential is periodic if and only if frequency f_0 is rational

Fourier series: Just like in CT, we want to express periodic $x[n]$ as a sum of the harmonic sinusoids $\phi_k[n]$

$$x[n] = \sum_k a_k e^{j2\pi(kf_0)n} = \sum_k a_k e^{j2\pi\left(\frac{k}{N_0}\right)n}$$

In CT we sum over $-\infty < k < \infty$

But in DT, unique frequencies are limited!

$$\underline{0 \leq f < 1}$$

$f = \frac{k}{N_0} \rightarrow k = 0, 1, 2, \ldots, N_0 - 1$ $\qquad$ If $k = N_0$, we get aliasing as $f = 1$

Same result if we add N_0 to any k

$$\rightarrow \phi_{k+N_0}[n] = e^{j2\pi\left(\frac{k+N_0}{N_0}\right)n} = e^{j2\pi\left(\frac{k}{N_0}\right)n} \underbrace{e^{j2\pi\frac{N_0}{N_0}n}}_{=1} = e^{j2\pi\left(\frac{k}{N_0}\right)n} = \phi_k[n] = \phi_{k+N_0}[n]$$

$\rightarrow$ We have N_0 unique harmonics, then they start to repeat

Only need N_0 harmonics to reconstruct signal!

$$\rightarrow x[n] = \sum_{k=0}^{N_0-1} a_k e^{j2\pi\left(\frac{k}{N_0}\right)n}$$

Just like in CT, we have a sum of harmonic frequencies of the fundamental frequency, with the k^{th} harmonic weighted by constant a_k

Finding a_k? Projection/Dot Product

DT dot product: $\langle a[n], b[n] \rangle = \sum a[n] b^*[n]$

$$\rightarrow a_k = \langle x[n], e^{j2\pi\left(\frac{k}{N_0}\right)n} \rangle = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-j2\pi\left(\frac{k}{N_0}\right)n}$$

Normalize by N_0 similar to $1/T_0$ in CT Fourier series

$$\boxed{\begin{aligned} &\underline{\text{DT Fourier Series Pair}} \\ &x[n] = \sum_{k=\langle N_0 \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N_0 \rangle} a_k e^{j2\pi\left(\frac{k}{N_0}\right)n} \\ &a_k = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x[n] e^{-j2\pi\left(\frac{k}{N_0}\right)n} \end{aligned}}$$

a_k's called spectral coefficients

$k = \langle N_0 \rangle$? Recall: harmonics repeat $\rightarrow \phi_k[n] = \phi_{k+N_0}[n] \rightarrow \boxed{a_k = a_{k+N_0}}$

DT FS coefficients are periodic w/ period N_0!

The DT FS sums above may be computed using any consecutive set of N_0 samples in n or k

Properties of DT FS

Symmetry: Same symmetry properties of CT FS

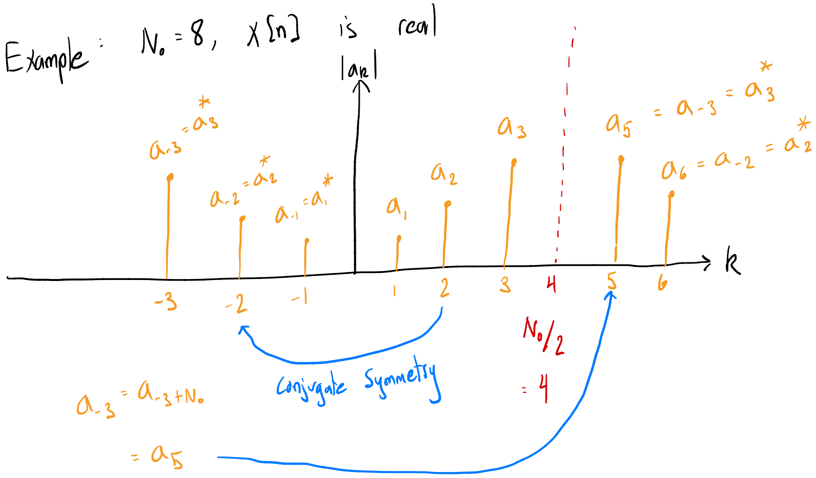
- If $x[n]$ is real, then $a_k = a_{-k}^*$ $\rightarrow$ $|a_k| = |a_{-k}|$ (even) $\quad \angle a_k = -\angle a_{-k}$ (odd)
- If $x[n]$ is real & even, then $a_k = a_{-k}^* = a_{-k}$
  - $\rightarrow$ a_k is real and even
- If $x[n]$ is real & odd, then $a_k = a_{-k}^* = -a_{-k}$
  - $\rightarrow$ a_k is imaginary and odd

For DT FS, $a_k = a_{k+N_0}$

$\rightarrow$ For real $x[n]$, $a_k = a_{-k}^* = a_{-k+N_0}^*$

$\rightarrow$ There is conjugate symmetry about $k = \frac{N_0}{2}$

Example: $N_0 = 8$, $x[n]$ is real



$\rightarrow$

a_0

$a_1 = a_{-1}^* = a_7^*$

$a_2 = a_{-2}^* = a_6^*$

$a_3 = a_{-3}^* = a_5^*$

$a_4 = a_{-4}^* = a_4^*$ (i.e., a_4 must be real)

$a_5 = a_{-5}^* = a_3^*$

$a_6 = a_{-6}^* = a_2^*$

$a_7 = a_1^* = a_{-1}^*$

$\rightarrow$ For real $x[n]$, only need $0 \le k \le \frac{N_0}{2}$ to uniquely specify signal

Parseval's Relation

Average Power: $\frac{1}{N_0}\sum_{n=\langle N_0\rangle}|x[n]|^2 = \sum_{k=\langle N_0\rangle}|a_k|^2$

Time - Shifting

If $x[n] \xleftrightarrow{FS} a_k$, then $x[n-n_0] \xleftrightarrow{FS} a_k e^{-j2\pi\left(\frac{k}{N_0}\right)n_0}$

Conjugation

If $x[n] \xleftrightarrow{FS} a_k$, then $x^*[n] \xleftrightarrow{FS} a_{-k}^*$

Time - Reversal

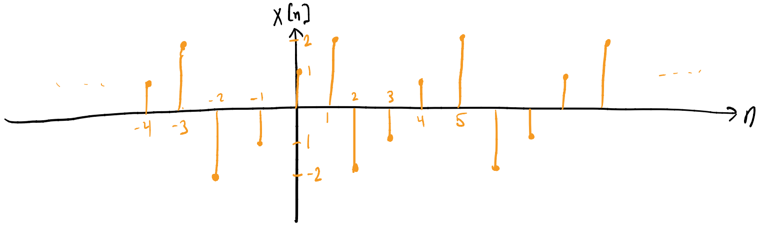
If $x[n] \xleftrightarrow{FS} a_k$, then $x[-n] \xleftrightarrow{FS} a_{-k}$

Periodic Convolution

If $x[n] \xleftrightarrow{FS} a_k$ and $y[n] \xleftrightarrow{FS} b_k$

then $\sum_{k=\langle N_0\rangle} x[k]\,y[n-k] \xleftrightarrow{FS} N_0 a_k b_k$

Example:



Find a_k. $\quad N_0 = 4$

$$\rightarrow a_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]\, e^{-j2\pi\left(\frac{k}{N_0}\right)n} = \frac{1}{4}\sum_{n=0}^{3} x[n]\, e^{-j\frac{\pi}{2}kn}$$

$$= \frac{1}{4}\left(x[0] + x[1]e^{-j\frac{\pi}{2}k} + x[2]e^{-j\pi k} + x[3]e^{-j\frac{3\pi}{2}k} \right)$$

$$= \frac{1}{4}\left(1 + 2e^{-j\frac{\pi}{2}k} - 2e^{-j\pi k} - e^{-j\frac{3\pi}{2}k} \right)$$

$k=0:\quad a_0 = \frac{1}{4}(1+2-2-1) = 0$

$k=1:\quad a_1 = \frac{1}{4}\left(1+2(-j)-2(-1)-(j)\right) = \frac{1}{4}(3-3j) = \frac{3}{4} - j\frac{3}{4}$

$k=2:\quad a_2 = \frac{1}{4}\left(1+2(-1)-2(1)-(-1)\right) = \frac{1}{4}(-2) = -\frac{1}{2}$

$k=3:\quad a_3 = \frac{1}{4}\left(1+2(j)-2(-1)-(-j)\right) = \frac{1}{4}(3+3j) = \frac{3}{4} + j\frac{3}{4}$

* Note: $a_3 = a_{-3}^* = a_1^*$

Example:

Real periodic $x[n]$ w/ period $N_0 = 10$

$a_0 = 0 \quad a_1 = 1 \quad a_2 = 0 \quad a_3 = ? \quad a_4 = ? \quad a_5 = 0 \quad a_6 = 0 \quad a_7 = 2j \quad a_8 = 0 \quad a_9 = ?$

Missing coefficients? Use symmetry: $a_k = a_{-k}^* = a_{-k+N_0}^*$

$\rightarrow a_3 = a_{-3}^* = a_{-3+N_0}^* = a_7^* = -2j$

$a_4 = a_{-4}^* = a_{-4+N_0}^* = a_6^* = 0$

$a_9 = a_{-9}^* = a_{-9+N_0}^* = a_1^* = 1$

So: $a_3 = a_7^* = -2j$

$a_9 = a_1^* = 1$

$a_0 = a_2 = a_4 = a_5 = a_6 = a_8 = 0$

Find $x[n]$: $\quad x[n] = \sum_{k=0}^{N_0-1} a_k e^{j2\pi\left(\frac{k}{N_0}\right)n} = \sum_{k=0}^{9} a_k e^{j\frac{\pi}{5}kn}$

$$= a_3 e^{j\frac{3\pi}{5}n} + a_7 e^{j\frac{7\pi}{5}n} + a_1 e^{j\frac{\pi}{5}n} + a_9 e^{j\frac{9\pi}{5}n}$$

$$= -2j e^{j\frac{3\pi}{5}n} + 2j \underbrace{e^{j\frac{7\pi}{5}n}}_{= e^{-j\frac{3\pi}{5}n}} + e^{j\frac{\pi}{5}n} + \underbrace{e^{j\frac{9\pi}{5}n}}_{= e^{-j\frac{\pi}{5}n}}$$

$$= -2j\left(\underbrace{e^{j\frac{3\pi}{5}n} - e^{-j\frac{3\pi}{5}n}}_{= 2j\sin\left(\frac{3\pi}{5}n\right)}\right) + \left(\underbrace{e^{j\frac{\pi}{5}n} + e^{-j\frac{\pi}{5}n}}_{= 2\cos\left(\frac{\pi}{5}n\right)}\right)$$

$$= \underline{4\sin\left(\frac{3\pi}{5}n\right) + 2\cos\left(\frac{\pi}{5}n\right)}$$