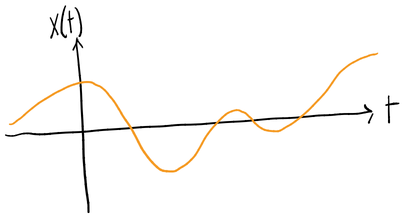
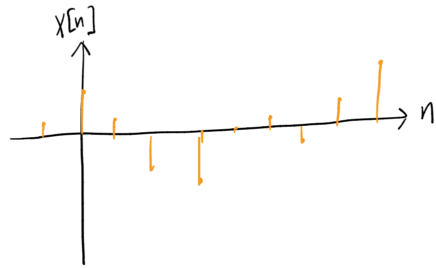
Discrete-Time Signals

In CT, we use $t \to x(t)$ — t is continuous

In DT, we use $n \to x[n]$ — n may only take integer values!





Unit spacing between stems

$x[n]$ undefined for non-integer n

Why DT?

- Digital computers process discrete data
- Can't perfectly replicate continuous phenomena (finite sampling rates)

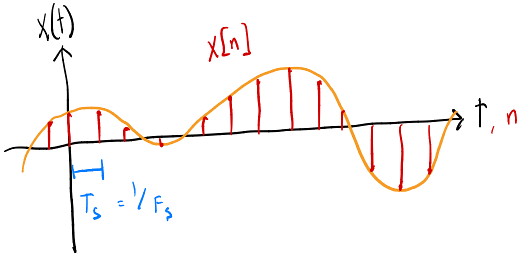
In ECE, DT signals usually just sampled representations of CT phenomena

e.g., Analog-to-Digital Conversion (ADC) of a continuous voltage signal

For a sampled signal:

$$x[n] = x(t)\Big|_{t = nT_s} = x(nT_s) \rightarrow T_s \triangleq \text{"Sampling Period"}$$

$$1/T_s = F_s \triangleq \text{"Sampling Frequency"}$$



All CT concepts have analogous concepts in DT:

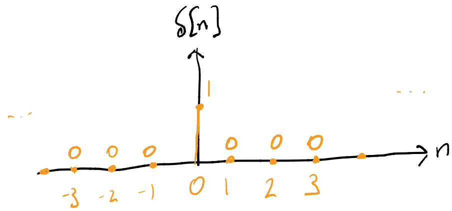
- CT convolution $\rightarrow$ DT convolution
- CT Fourier Series $\rightarrow$ DT Fourier Series (DTFS)
- CT Fourier Transform $\rightarrow$ DT Fourier Transform (DTFT)
- Laplace Transform $\rightarrow$ z - Transform

Discrete-Time Convolution

Recall:

Impulse (Delta) function:

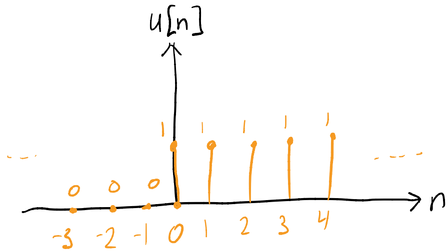
$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{else} \end{cases}$$



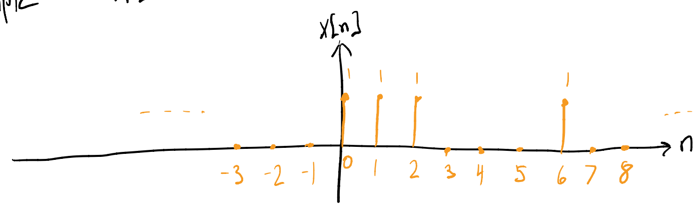
Step function:

Note ≥

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{else} \end{cases}$$



Example: $x[n] = u[n] - u[n-3] + \delta[n-6]$



$x[n]$ can be written as a sum of impulses: $x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-6]$

More complex $x[n]$? e.g., $x_2[n] = (0.5)^n u[n]$

$\rightarrow$ Still works! $x_2[n] = \delta[n] + 0.5\,\delta[n-1] + 0.25\,\delta[n-2] + \dots$

$$= \sum_{k=0}^{\infty} (0.5)^k \delta[n-k] = \sum_{k=0}^{\infty} x_2[k]\,\delta[n-k]$$

So... DT signals are sums of scaled, shifted impulses: $x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$

Similar to CT

$\rightarrow x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$

Recall LTI systems: work the same in DT as in CT

LTI System

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Input

DT impulse response

Output

$$h[n]: \quad \delta[n] \longrightarrow \boxed{H} \longrightarrow h[n]$$

By definition: $x[n] = \delta[n] \xrightarrow{H} y[n] = h[n]$

Time-Invariance: $\delta[n-n_0] \xrightarrow{H} h[n-n_0]$

Linearity: $a_0\,\delta[n-n_0] + a_1\,\delta[n-n_1] \xrightarrow{H} a_0\,h[n-n_0] + a_1\,h[n-n_1]$

Finally: $x[n] = \dots + x[-1]\,\delta[n+1] + x[0]\,\delta[n] + x[1]\,\delta[n-1] + \dots$

$$= \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \xrightarrow{H} \boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]}$$

DT Convolution Sum $\rightarrow$ Output of DT LTI system w/ input $x[n]$ & impulse response $h[n]$

Similarity to CT convolution integral:

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$ ← Integral because domain is continuous

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h(n-k)$$ ← Sum because n must be an integer

Convolution Properties:

a.) Commutative Property: $x_1[n] * x_2[n] = x_2[n] * x_1[n]$

b.) Distributive Property: $x_1[n] * (x_2[n] + x_3[n]) = x_1[n] * x_2[n] + x_1[n] * x_3[n]$

c.) Associative Property: $(x_1[n] * x_2[n]) * x_3[n] = x_1[n] * (x_2[n] * x_3[n])$

d.) Shifting Property: If $x_1[n] * x_2[n] = y[n] \rightarrow x_1[n-k] * x_2[n-p] = y[n-k-p]$

e.) Convolution w/ Impulse: $x[n] * \delta[n] = x[n] \rightarrow \underline{x[n] * \delta[n-n_0] = x[n-n_0]}$

f.) Width of Output: If $x_1[n]$ has width L_1 samples & $x_2[n]$ has width L_2 samples

$\rightarrow y[n] = x_1[n] * x_2[n]$ has width $L_1 + L_2 - 1$ samples

$x_1[n]$ "turns on" at $n = n_1$

$x_2[n]$ "turns on" at $n = n_2$

$\rightarrow y[n] = x_1[n] * x_2[n]$ "turns on" at $n = n_1 + n_2$

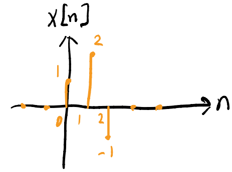
Similarly for when signals "turn off"

g.) Causal systems: $h[n] = 0$ for $n < 0 \rightarrow y[n] = x[n] * h[n] = \sum_{k=0}^{\infty} x[k] h[n-k]$

h.) BIBO Stable systems: $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

Visual Example:

$$x[n] = \delta[n] + 2\delta[n-1] - \delta[n-2]$$



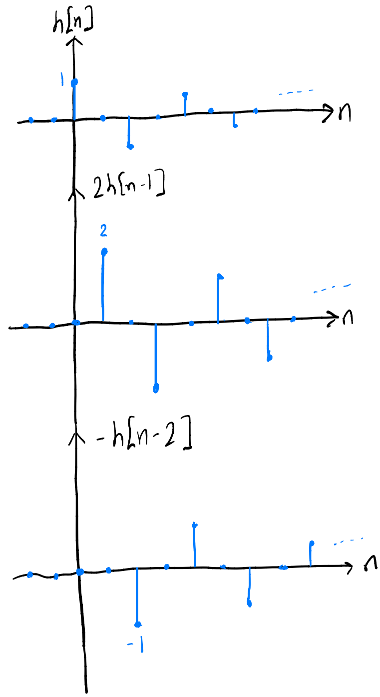
$$h[n] = (0.9)^n \cos\left(\frac{\pi}{2}n\right) u[n]$$

Each impulse in $x[n]$ produces a scaled, shifted copy of $h[n]$...

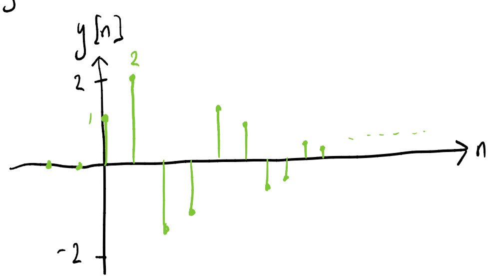
$\delta[n] \rightarrow h[n]$

$2\delta[n-1] \rightarrow 2h[n-1]$

$-\delta[n-2] \rightarrow -h[n-2]$



Finally, $y[n] = h[n] + 2h[n-1] - h[n-2]$



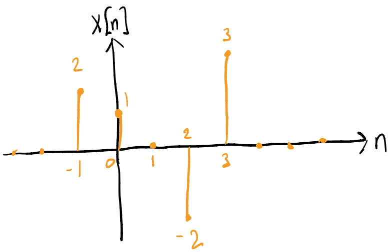
DT Convolution Methods

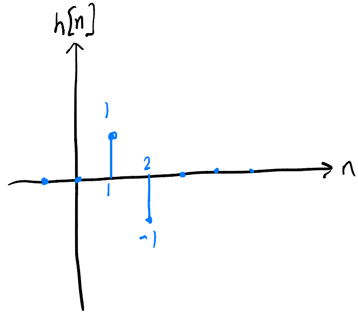
1. Graphical "flip and shift" method similar to CT convolution
2. For finite sequences $x[n]$ & $h[n]$, can "brute force" the convolution sum

"Brute Force" Example:

$$x[n] = 2\delta[n+1] + \delta[n] - 2\delta[n-2] + 3\delta[n-3]$$

$$h[n] = \delta[n-1] - \delta[n-2]$$





Convolution Sum: $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$

* In "brute-force" convolution, it is easier to "flip-and-shift" the longer sequence — $x[n]$ in this case

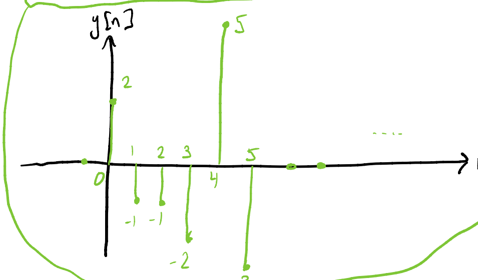
→ $h[k]$ only non-zero for $k=1$ & $k=2$

→ $y[n] = \sum_{k=1}^{2} h[k]\,x[n-k] = h[1]\,x[n-1] + h[2]\,x[n-2] = \underline{x[n-1] - x[n-2]}$

Make a table:

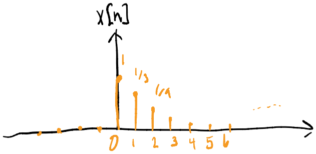
n	-2	-1	0	1	2	3	4	5	6
$x[n-1]$	0	0	2	1	0	-2	3	0	0
$-x[n-2]$	0	0	0	-2	-1	0	2	-3	0
$y[n]$ (Sum)	0	0	2	-1	-1	-2	5	-3	0

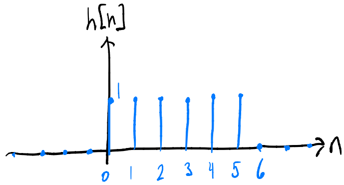
$$\boxed{y[n] = 2\delta[n] - \delta[n-1] - \delta[n-2] - 2\delta[n-3] + 5\delta[n-4] - 3\delta[n-5]}$$



Flip and shift example:

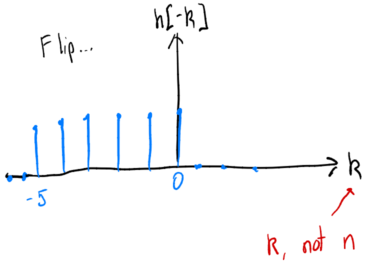
$$x[n] = \left(\tfrac{1}{3}\right)^n u[n]$$
$$h[n] = u[n] - u[n-6]$$



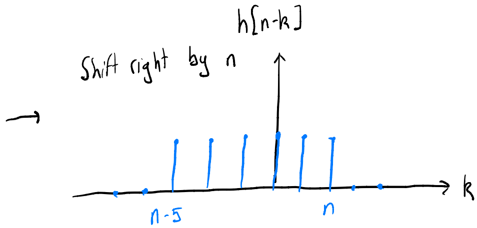


Easiest now to flip and shift $h[n] \rightarrow h[n-k] = h[-(k-n)]$

Flip...

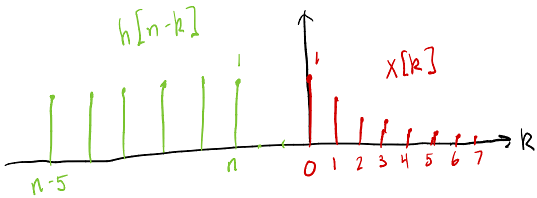


Shift right by n



Region 1: $n < 0 \rightarrow$ no overlap

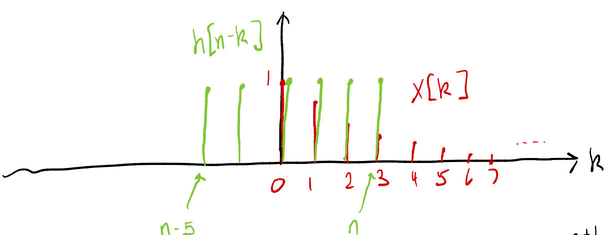
$\rightarrow y[n] = 0$



Just like CT convolution, we look for regions w/ different overlap such that product $x[k]h[n-k]$ or $h[k]x[n-k]$ is not zero, then evaluate summation

Region 2: Partial overlap from $k=0$ to $k=n$ $\rightarrow$ valid for $0 \le n \le 5$

$$\rightarrow y[n] = \sum_{k=0}^{n} 1 \cdot \left(\tfrac{1}{3}\right)^k = \sum_{k=0}^{n} \left(\tfrac{1}{3}\right)^k$$



Summation formula (on eq. sheet):

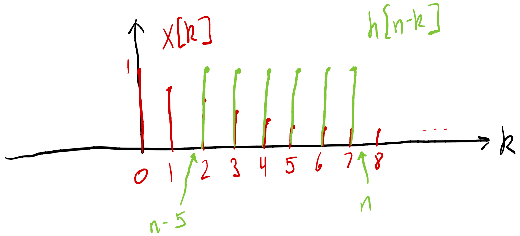
$$\sum_{k=n_0}^{n_1} a^k = \frac{a^{n_0} - a^{n_1+1}}{1-a}$$

$a = \frac{1}{3}$

$n_0 = 0,\ n_1 = n$

$$\rightarrow y[n] = \frac{a^0 - a^{n+1}}{1-a} = \frac{1 - \left(\frac{1}{3}\right)^{n+1}}{2/3} = \frac{3}{2}\left(1 - \left(\tfrac{1}{3}\right)^{n+1}\right) = \boxed{\frac{3}{2}\left(1 - \tfrac{1}{3}\left(\tfrac{1}{3}\right)^n\right) \quad 0 \le n \le 5}$$

Region 3: Full overlap → valid for $n > 5$



Full overlap from $k = n-5$ to $k = n$

$$\rightarrow y[n] = \sum_{k=n-5}^{n} 1 \cdot \left(\tfrac{1}{3}\right)^k = \sum_{k=n-5}^{n} \left(\tfrac{1}{3}\right)^k$$

$a = \frac{1}{3}$, $n_0 = n-5$, $n_1 = n$

$$\rightarrow y[n] = \frac{a^{n-5} - a^{n+1}}{1-a} = \frac{\left(\frac{1}{3}\right)^{n-5} - \left(\frac{1}{3}\right)^{n+1}}{1 - \frac{1}{3}} = \frac{\left(\frac{1}{3}\right)^{-5}\left(\frac{1}{3}\right)^{n} - \frac{1}{3}\left(\frac{1}{3}\right)^{n}}{2/3} = \frac{\left(3^5 - \frac{1}{3}\right)\left(\frac{1}{3}\right)^n}{2/3}$$

$$= \frac{3}{2}\,\frac{728}{3}\left(\tfrac{1}{3}\right)^n = \boxed{364\left(\tfrac{1}{3}\right)^n \quad n > 5}$$

$$\rightarrow \boxed{y[n] = \begin{cases} 0 & n < 0 \\ \frac{3}{2}\left(1 - \frac{1}{3}\left(\frac{1}{3}\right)^n\right) & 0 \le n \le 5 \\ 364\left(\frac{1}{3}\right)^n & 5 < n \end{cases}}$$

Note: at $n = 5$, the neighboring expressions are equal!

$\frac{3}{2}\left(1 - \frac{1}{3}\left(\frac{1}{3}\right)^5\right) = 364\left(\frac{1}{3}\right)^5 \approx 1.4979$ ✓

Check the boundary points to see if answers line up!