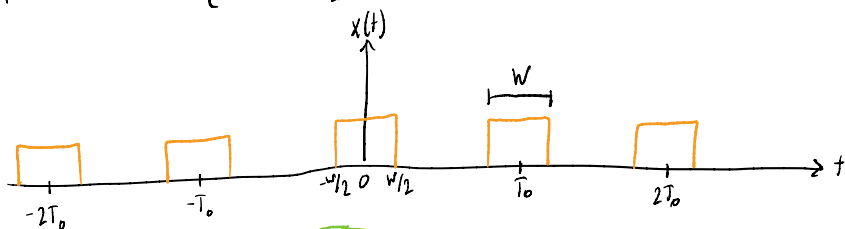


The Fourier Transform

FS example: $x(t) = \begin{cases} 1 & |t| < W/2 \\ 0 & W/2 < |t| < T_0/2 \end{cases}$



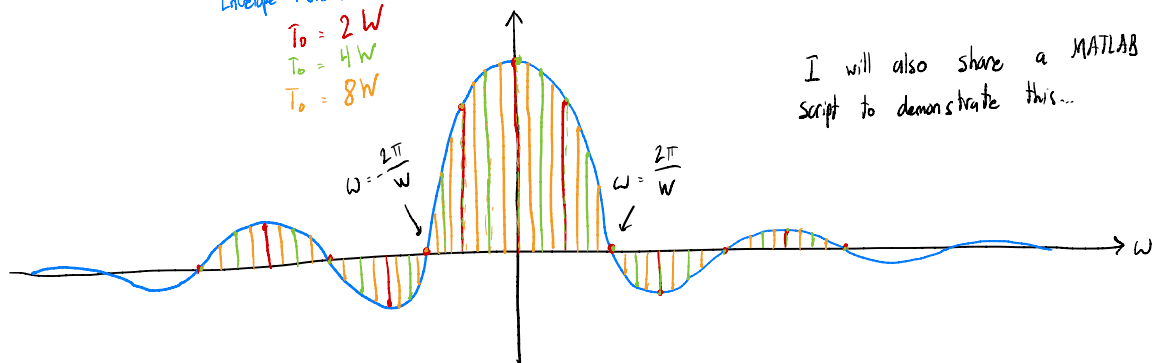
Result from HW: $a_k = \frac{2 \sin(k \omega_0 \frac{W}{2})}{k \omega_0 T_0}$

Suppose we hold W constant (rectangle width does not change) but increase T_0 /decrease ω_0

We can rewrite like this: $T_0 a_k = \frac{2 \sin(\frac{\omega W}{2})}{\omega} \Big|_{\omega = k \omega_0}$

Annotations:
 - $T_0 a_k$ is labeled "Scaled coefficients"
 - $\frac{2 \sin(\frac{\omega W}{2})}{\omega}$ is labeled "Envelope function Constant W/T_0 !" and "Continuous"
 - $\omega = k \omega_0$ is labeled "Sampled at integer multiples of ω_0 "

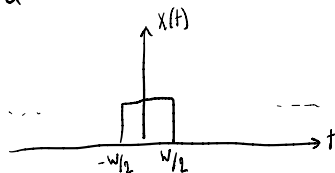
Envelope Function
 $T_0 = 2W$
 $T_0 = 4W$
 $T_0 = 8W$



I will also share a MATLAB script to demonstrate this...

Result: As T_0 increases, the a_k values (scaled by T_0) become more closely spaced

Q: What if $T_0 \rightarrow \infty$?

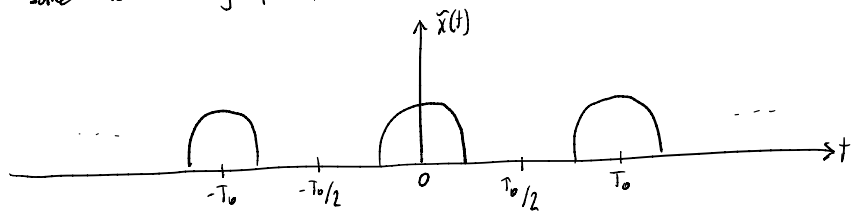


A: $x(t)$ becomes aperiodic and the FS coefficients become arbitrarily close to each other, approaching the continuous envelope function

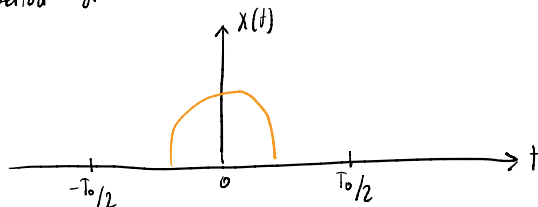
Okay, so a_k approaches a continuous envelope function as $T_0 \rightarrow \infty$ and $X(t) \rightarrow$ aperiodic.

What is this envelope function?

Let's define some new arbitrary periodic function $\tilde{X}(t)$



Let $x(t)$ be just one period of this function:



That is,
$$x(t) = \begin{cases} \tilde{X}(t) & -T_0/2 < t < T_0/2 \\ 0 & \text{else} \end{cases}$$

The FS equations: $\star \tilde{X}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ and $a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \tilde{X}(t) e^{-jk\omega_0 t} dt$

Because $x(t) = \tilde{X}(t)$ for $-T_0/2 < t < T_0/2$ and $= 0$ elsewhere, can also write:

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T_0} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt$$

Remember the envelope function above is the envelope of $T_0 a_k$, not a_k

$\rightarrow$ Envelope $\triangleq X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ and $a_k = \frac{1}{T_0} X(k\omega_0) \star$

Combining $\star$ and $\star$ obtains: $\tilde{X}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} X(k\omega_0) e^{jk\omega_0 t}$

Note that $\frac{1}{T_0} = \frac{\omega_0}{2\pi} \rightarrow \tilde{X}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\omega_0) e^{jk\omega_0 t} \omega_0$

As $T_0 \rightarrow \infty$, $\tilde{x}(t) \rightarrow x(t)$; $\omega_0 \rightarrow 0$ (a differential $d\omega$!); $k\omega_0 \rightarrow \omega$ (continuous)

→ Summation becomes an integral

$$\rightarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \quad \text{Inverse Fourier Transform}$$

and from above:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \text{Fourier Transform}$$

Compare to Fourier Series:

• Synthesis: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

vs. $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$

- Both expressions allow us to express signals as a linear combination of sinusoids at different frequencies

- For periodic $x(t)$, sinusoids have amplitudes a_k and occur only at harmonically related frequencies $k\omega_0$, $\omega = \pm\omega_0, \pm2\omega_0, \dots$
- For aperiodic $x(t)$, sinusoids have "amplitude" $\frac{1}{2\pi} X(\omega) d\omega$ and occur over a continuum of frequencies, $\omega \in (-\infty, \infty)$
- a_k called "spectral coefficients," $X(\omega)$ called "spectrum"

• Analysis: $a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$ vs. $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

- Both Fourier representations of signals are projections (dot product) of $x(t)$ with complex sinusoids

- Periodic: projection onto harmonic sinusoid
- Aperiodic: projection onto continuum of all possible sinusoids

Fourier Transform Pair:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Also: $\omega = 2\pi F \rightarrow$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

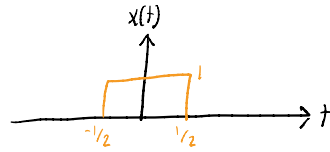
$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF$$

Note: the book only uses ω , but we will frequently use F owing to its simplicity
 You may use either ω or F on assignments/exams, and both are on the eq. sheet

Example:

$$x(t) = \text{rect}(t) = \begin{cases} 0 & |t| > 1/2 \\ 1 & |t| < 1/2 \end{cases}$$



FT operator

Find $\mathcal{F}\{x(t)\} = X(F)$.

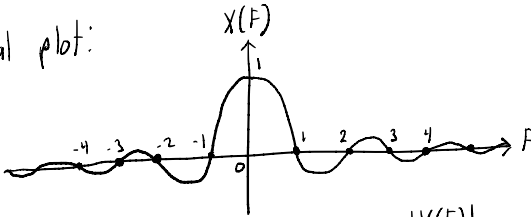
$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt = \int_{-1/2}^{1/2} (1) e^{-j2\pi Ft} dt = -\frac{1}{j2\pi F} e^{-j2\pi Ft} \Big|_{-1/2}^{1/2}$$

$$= \frac{1}{j2\pi F} (e^{-j\pi F} - e^{j\pi F}) = \frac{1}{j2\pi F} (\underbrace{e^{-j\pi F} - e^{j\pi F}}_{= j2 \sin(\pi F)}) = \frac{j2 \sin(\pi F)}{j2\pi F}$$

$$= \frac{\sin(\pi F)}{\pi F}$$

$$\frac{\sin(\pi F)}{\pi F} \triangleq \text{sinc}(F)$$

All real plot:

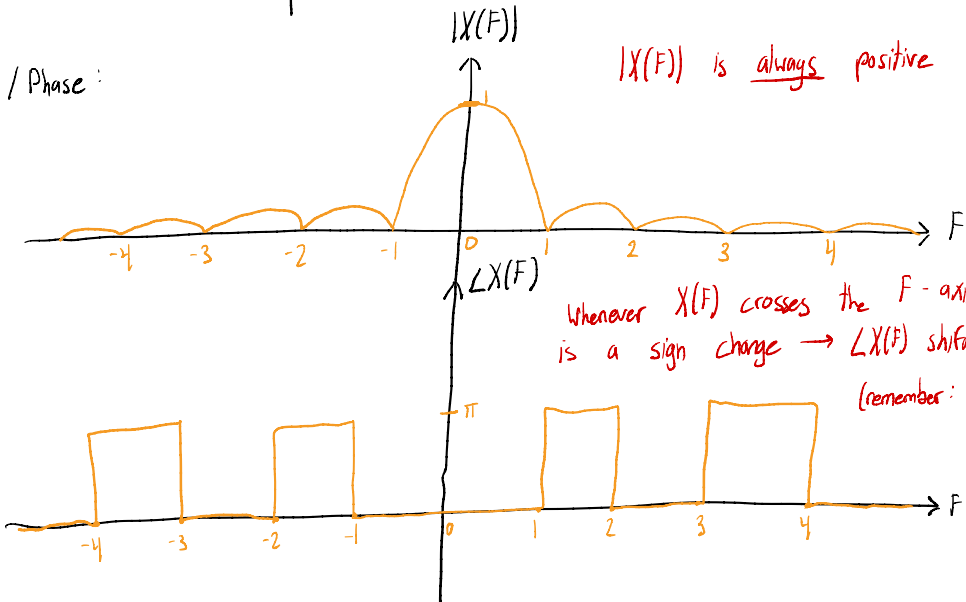


@ $F=0 \rightarrow \text{peak}, X(F)=1$

@ $F = \pm 1, \pm 2, \dots \rightarrow X(F) = 0$

Mag. / Phase:

$|X(F)|$ is always positive

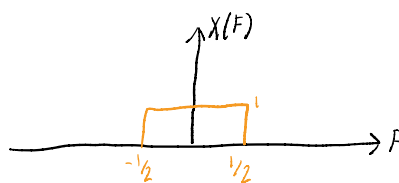


Whenever $X(F)$ crosses the F -axis, there is a sign change $\rightarrow L(X(F))$ shifts by $\pm \pi$
 (remember: $e^{\pm j\pi} = -1$!)

What about the other way?

$$X(F) = \text{rect}(F)$$

$$\text{Find } \mathcal{F}^{-1}\{X(F)\} = x(t).$$



$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF = \int_{-1/2}^{1/2} (1) e^{j2\pi Ft} dF = \left. \frac{1}{j2\pi t} e^{j2\pi Ft} \right|_{-1/2}^{1/2} \\ &= \frac{1}{j2\pi t} \left(\underbrace{e^{j\pi t} - e^{-j\pi t}}_{= j2\sin(\pi t)} \right) = \frac{j2\sin(\pi t)}{j2\pi t} = \frac{\sin(\pi t)}{\pi t} = \boxed{\text{sinc}(t)} \end{aligned}$$

• A rect function in either domain produces a sinc function in the other domain!

• There is a duality in Fourier analysis

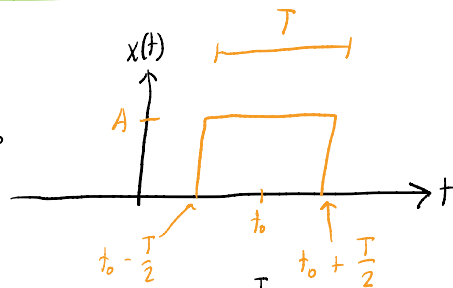
- Sometimes, we can reverse the roles of time and frequency

- If one operation in the time domain produces a change in the frequency domain, then a similar operation in the frequency domain produces an analogous change in the time domain

More general rect function:

$$x(t) = A \text{rect}\left(\frac{t-t_0}{T}\right)$$

- Scaled by A in amplitude
- Center of rect shifted to $t = t_0$
- Rect width = T



Find $X(F)$.

$$\begin{aligned} X(F) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt = \int_{t_0 - T/2}^{t_0 + T/2} (A) e^{-j2\pi Ft} dt = \left. \frac{A}{-j2\pi F} e^{-j2\pi Ft} \right|_{t_0 - T/2}^{t_0 + T/2} \\ &= \frac{A}{-j2\pi F} \left(e^{-j2\pi F(t_0 + T/2)} - e^{-j2\pi F(t_0 - T/2)} \right) = \frac{A}{-j2\pi F} \left(\underbrace{e^{-j2\pi Ft_0}}_{\text{common factor}} \underbrace{e^{-j\pi FT}}_{\text{common factor}} \left(e^{-j\pi F T/2} - e^{j\pi F T/2} \right) \right) \\ &= \frac{A}{j2\pi F} e^{-j2\pi Ft_0} \left(\underbrace{e^{j\pi FT} - e^{-j\pi FT}}_{= j2\sin(\pi FT)} \right) = \frac{j2A\sin(\pi FT)}{j2\pi F} e^{-j2\pi Ft_0} \end{aligned}$$

$$= A \underbrace{T}_{\text{sinc}(FT)} \frac{\sin(\pi FT)}{\pi FT} e^{-j2\pi Ft_0} = \boxed{AT \text{sinc}(FT) e^{-j2\pi Ft_0}}$$

- Scaled by A : Multiplying $x(t)$ by A results in multiplying $X(F)$ by A
- We would get the same result from $\mathcal{F}\{x_1(t) + x_2(t)\} = X_1(F) + X_2(F)$
- Property: The Fourier Transform is linear $\rightarrow \mathcal{F}\{a_1 x_1(t) + a_2 x_2(t)\} = a_1 X_1(F) + a_2 X_2(F)$

- Amplitude and F

scaled by T :

- Scaling time-axis results in opposite scaling of frequency-axis
- "speeding up" in time leads to broadened $X(F)$; "slowing down" in time leads to a compressed $X(F)$ $\rightarrow$ Faster functions require more/higher frequencies
- Also see a corresponding amplitude change related to T

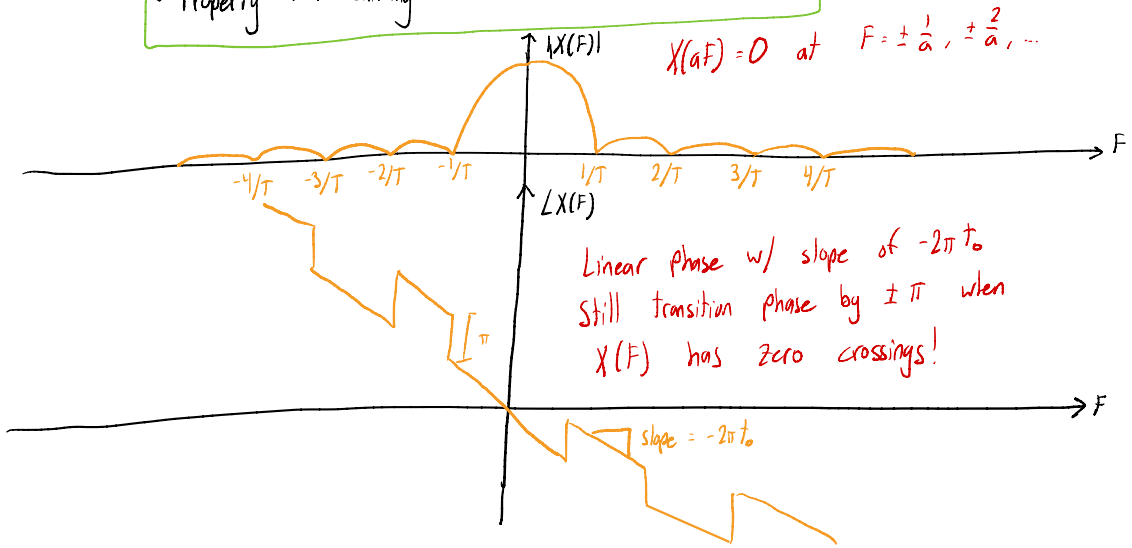
Property: Time/Frequency-scaling $\rightarrow \mathcal{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{F}{a}\right)$

- Linear phase

shift $e^{-j2\pi Ft_0}$:

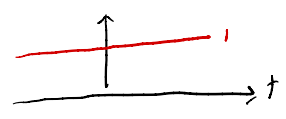
- Delaying in time by t_0 does not impact amplitude of $X(F) \rightarrow |X(F)|$ unchanged
- Linear phase shift by $-2\pi Ft_0$ over all F

Property: Time-shifting $\rightarrow \mathcal{F}\{x(t-t_0)\} = X(F) e^{-j2\pi Ft_0}$



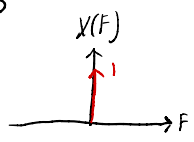
Example: $x(t) = 1$ Find $X(F)$.

$$X(F) = \int_{-\infty}^{\infty} (1) e^{-j2\pi Ft} dt$$



→ For any value $F \neq 0$, we integrate a sinusoid over all time → averages to 0

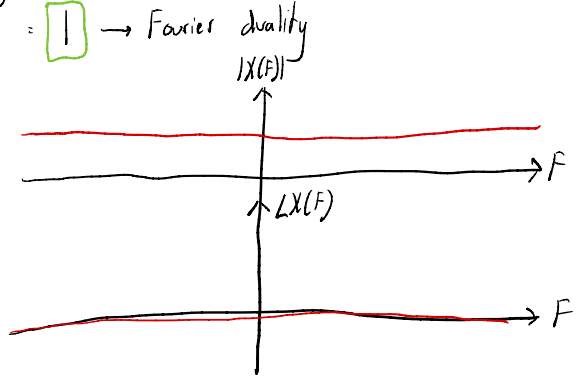
But at $F=0$: $X(F) = \int_{-\infty}^{\infty} (1) dt = \infty \rightarrow X(F) = \begin{cases} \infty & F=0 \\ 0 & \text{else} \end{cases} = \boxed{\delta(F)}$



Other way around? $x(t) = \delta(t)$ Find $X(F)$.

$$X(F) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi Ft} dt = e^{-j2\pi F(0)} = \boxed{1} \rightarrow \text{Fourier duality}$$

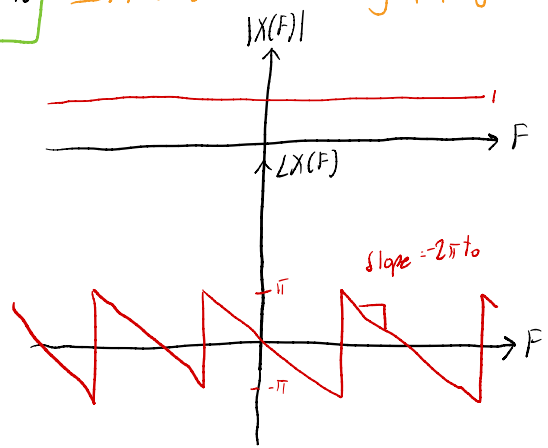
Sifting property:
 $\int_{-\infty}^{\infty} \delta(t) x(t) dt = x(0)$



Delayed Delta? $x(t) = \delta(t - t_0)$ Find $X(F)$.

$$X(F) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi Ft} dt = \boxed{e^{-j2\pi Ft_0}} \rightarrow \text{Matches the time-delay property!}$$

Sifting property



Note: in your own phase plots, you may or may not wrap to within $[-\pi, \pi]$

Example: $x(t) = \cos(2\pi F_0 t)$ Find $X(F)$.

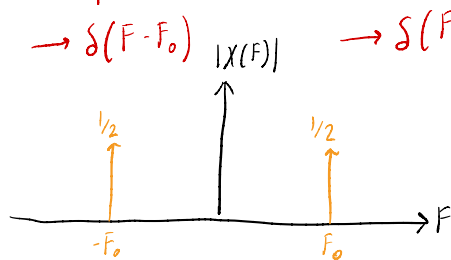
Recall: $\cos(2\pi F_0 t) = \frac{1}{2} e^{j2\pi F_0 t} + \frac{1}{2} e^{-j2\pi F_0 t}$

$$\rightarrow X(F) = \int_{-\infty}^{\infty} \left(\frac{1}{2} e^{j2\pi F_0 t} + \frac{1}{2} e^{-j2\pi F_0 t} \right) e^{-j2\pi F t} dt = \frac{1}{2} \int_{-\infty}^{\infty} e^{-j2\pi (F-F_0)t} dt + \frac{1}{2} \int_{-\infty}^{\infty} e^{-j2\pi (F+F_0)t} dt$$

Integrates to 0 except when $F = F_0$
Integrates to 0 except when $F = -F_0$

$\rightarrow \delta(F-F_0)$
 $\rightarrow \delta(F+F_0)$

$$= \frac{1}{2} \delta(F-F_0) + \frac{1}{2} \delta(F+F_0)$$



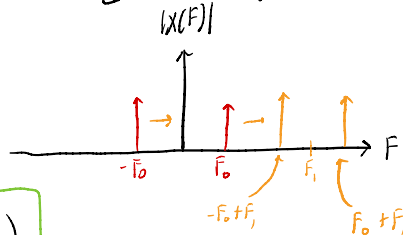
By similar reasoning, we can show that

$$x(t) = e^{j2\pi F_0 t} \xleftrightarrow{\mathcal{F}} X(F) = \delta(F-F_0)$$

Example: $x(t) = \cos(2\pi F_0 t) e^{j2\pi F_1 t}$ Find $X(F)$.

Rewrite: $x(t) = \frac{1}{2} (e^{j2\pi F_0 t} + e^{-j2\pi F_0 t}) e^{j2\pi F_1 t} = \frac{1}{2} (e^{j2\pi (F_1+F_0)t} + e^{j2\pi (F_1-F_0)t})$

$$\rightarrow X(F) = \frac{1}{2} \delta(F-F_0-F_1) + \frac{1}{2} \delta(F+F_0-F_1)$$



Property: Frequency-shifting $\rightarrow \mathcal{F}\{x(t) e^{j2\pi F_0 t}\} = X(F-F_0)$

Let's revisit $x(t) = \text{Arect}\left(\frac{t-t_0}{T}\right) \xleftrightarrow{\mathcal{F}} AT \text{sinc}(FT) e^{-j2\pi F t_0}$

We can rewrite $x(t)$ as $x(t) = \underbrace{\text{Arect}\left(\frac{t}{T}\right)}_{x_1(t)} * \underbrace{\delta(t-t_0)}_{x_2(t)}$

$$\rightarrow x_1(t) = \text{Arect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} AT \text{sinc}(FT) = X_1(F)$$

$$\rightarrow x_2(t) = \delta(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j2\pi F t_0} = X_2(F)$$

$$\rightarrow \text{Arect}\left(\frac{t}{T}\right) * \delta(t-t_0) = x_1(t) * x_2(t) \xleftrightarrow{\mathcal{F}} AT \text{sinc}(FT) e^{-j2\pi F t_0} = X_1(F) X_2(F)$$

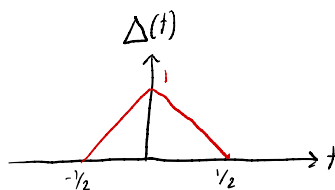
Property: Convolution $\rightarrow \mathcal{F}\{x_1(t) * x_2(t)\} = X_1(F) X_2(F)$

What about multiplication of $x_1(t) x_2(t)$? Fourier duality suggests convolution in F !

Property: Multiplication $\rightarrow \mathcal{F}\{x_1(t) x_2(t)\} = X_1(F) * X_2(F)$

Example: Triangle function

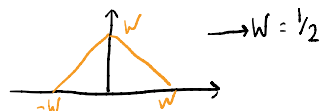
$$\Delta(t) = \begin{cases} 0 & |t| \geq \frac{1}{2} \\ 1 - 2|t| & |t| < \frac{1}{2} \end{cases}$$



Find $\mathcal{F}\{\Delta(t)\}$.

Could integrate... but remember from HW: triangle is the result of convolving 2 equal rect functions

$$\text{rect}(t/W) * \text{rect}(t/W) = \begin{cases} 0 & |t| > W \\ W - |t| & |t| \leq W \end{cases}$$



Linearity $\rightarrow \text{Arect}(t/W) * \text{Arect}(t/W) = \begin{cases} 0 & |t| > W \\ A^2(W - |t|) & |t| \leq W \end{cases} \rightarrow A^2 W = 1 \rightarrow A = \sqrt{2}$

$$\rightarrow \Delta(t) = \sqrt{2} \text{rect}(t/0.5) * \sqrt{2} \text{rect}(t/0.5) \xleftrightarrow{\mathcal{F}} \Delta(F) = \mathcal{F}\left\{\sqrt{2} \text{rect}(t/0.5)\right\} \mathcal{F}\left\{\sqrt{2} \text{rect}(t/0.5)\right\}$$

$$= \frac{\sqrt{2}}{2} \text{sinc}\left(\frac{F}{2}\right)$$

$$\rightarrow \Delta(F) = \frac{1}{2} \text{sinc}^2\left(\frac{F}{2}\right)$$

Quick Summary so far:

- We can compute FT directly

$$\rightarrow X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt \quad \text{and} \quad x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF$$

- But there is often a simpler way w/o integration

- We've learned a few basic FT pairs

$$- \text{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} T \text{sinc}(TF)$$

$$- W \text{sinc}(Wt) \xleftrightarrow{\mathcal{F}} \text{rect}\left(\frac{F}{W}\right)$$

$$- 1 \xleftrightarrow{\mathcal{F}} \delta(F)$$

$$- \delta(t) \xleftrightarrow{\mathcal{F}} 1$$

$$- e^{j2\pi F_0 t} \xleftrightarrow{\mathcal{F}} \delta(F - F_0)$$

$$- \cos(2\pi F_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2} \delta(F - F_0) + \frac{1}{2} \delta(F + F_0)$$

$$- \Delta(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2} \text{sinc}^2\left(\frac{F}{2}\right)$$

- We've also learned some relationships/properties:

- Linearity:

$$a_1 x_1(t) + a_2 x_2(t) \xleftrightarrow{\mathcal{F}} a_1 X_1(F) + a_2 X_2(F)$$

- Time/Frequency-Scaling:

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{F}{a}\right)$$

- Time-Shifting:

$$x(t - t_0) \xleftrightarrow{\mathcal{F}} X(F) e^{-j2\pi F t_0}$$

- Frequency-Shifting:

$$x(t) e^{j2\pi F_0 t} \xleftrightarrow{\mathcal{F}} X(F - F_0)$$

- Convolution:

$$x_1(t) * x_2(t) \xleftrightarrow{\mathcal{F}} X_1(F) X_2(F)$$

- Multiplication:

$$x_1(t) x_2(t) \xleftrightarrow{\mathcal{F}} X_1(F) * X_2(F)$$

→ Most often, it is easiest to apply these known pairs/properties in combination rather than using the integral form

→ There are a few more (all on your equation sheet), but let's do another example first

Example: $x(t) = 4000 \cos(600\pi t) \text{sinc}(200t)$

a.) Find $X(F)$.

This signal isn't on the equation sheet! But, parts of it are

$$\rightarrow W \text{sinc}(Wt) \xleftrightarrow{F} \text{rect}\left(\frac{F}{W}\right)$$

$$\rightarrow \cos(2\pi F_0 t) \xleftrightarrow{F} \frac{1}{2} \delta(F - F_0) + \frac{1}{2} \delta(F + F_0)$$

We can reorganize $x(t)$ to be in terms of these pairs:

$$x(t) = \underbrace{(200 \text{sinc}(200t))}_{x_1(t)} \underbrace{(\cos(600\pi t))}_{x_2(t)} \cdot 20 \quad (20 \cdot 200 = 4000)$$

$$= 20 x_1(t) x_2(t)$$

$$\rightarrow X_1(F) = \text{rect}\left(\frac{F}{200}\right)$$

$$\rightarrow X_2(F) = \frac{1}{2} \delta(F - 300) + \frac{1}{2} \delta(F + 300)$$

Okay, we have some more basic signal "building blocks" and their Fourier transforms.

We can combine them now using FT properties!

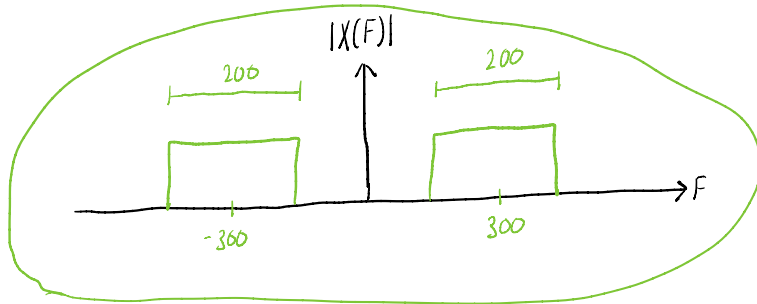
Property 1: Linearity $\rightarrow$ Multiply by 20 in time-domain means we also multiply by 20 in frequency domain!

Property 2: Multiplication $\rightarrow x_1(t) x_2(t) \xleftrightarrow{F} X_1(F) * X_2(F)$

$$\rightarrow X(F) = 20 \left(\text{rect}\left(\frac{F}{200}\right) * \left(\frac{1}{2} \delta(F - 300) + \frac{1}{2} \delta(F + 300) \right) \right)$$

$$\text{Remember: } X(F) * \delta(F - F_0) = X(F - F_0)$$

$$= 20 \left(\frac{1}{2} \text{rect}\left(\frac{F - 300}{200}\right) + \frac{1}{2} \text{rect}\left(\frac{F + 300}{200}\right) \right) = \boxed{10 \text{rect}\left(\frac{F - 300}{200}\right) + 10 \text{rect}\left(\frac{F + 300}{200}\right)}$$



Solving more complex Fourier transform problems:

- 1.) Break signals up into simple "building blocks" whose FT can be computed from a lookup on the "pairs" table, and compute their FTs.
- 2.) Determine how the "building blocks" are modified from their basic form (for instance, time-shifted, time-scaled, etc.), and apply changes to FTs based on a lookup from the "properties" table.
- 3.) Observe how the "building blocks" are related to one another (for instance, addition, multiplication, convolution), and combine the FTs together based on descriptions from the "properties" table.
- 4.) Make a plot at the end! In general, I will specify whether I want magnitude and/or phase, but if in doubt, plot both.

Example continued... $x(t) = 4000 \cos(600\pi t) \text{sinc}(200t) \xleftrightarrow{F} X(F) = 10 \text{rect}\left(\frac{F-300}{200}\right) + 10 \text{rect}\left(\frac{F+300}{200}\right)$

Suppose $y(t) = \underbrace{x(t-5)}_{\xleftrightarrow{F} X(F)} \underbrace{\cos(1600\pi t)}_{\xleftrightarrow{F} \frac{1}{2}\delta(F-800) + \frac{1}{2}\delta(F+800)}$ $\rightarrow$ Find $Y(F)$.

1.) Building blocks: $x(t) \xleftrightarrow{F} X(F)$
 $\cos(1600\pi t) \xleftrightarrow{F} \frac{1}{2}\delta(F-800) + \frac{1}{2}\delta(F+800)$

2.) $x(t)$ is modified to $x(t-5)$

Property: Time-delay $\rightarrow x(t-t_0) \xleftrightarrow{F} X(F) e^{-j2\pi F t_0}$

$$\rightarrow x(t-5) \xleftrightarrow{F} X(F) e^{-j10\pi F}$$

3.) Again we are multiplying in time, so convolving in frequency:

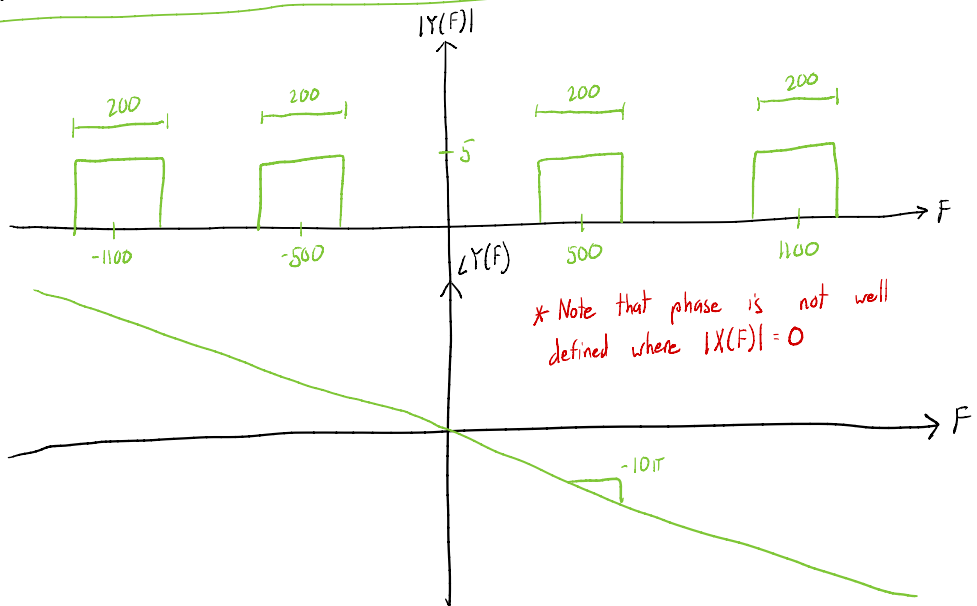
$$x(t-5) \cos(1600\pi t) \xleftrightarrow{F} (X(F) e^{-j10\pi F}) * \left(\frac{1}{2} \delta(F-800) + \frac{1}{2} \delta(F+800) \right)$$

$$\rightarrow Y(F) = \frac{1}{2} X(F-800) e^{-j10\pi(F-800)} + \frac{1}{2} X(F+800) e^{-j10\pi(F+800)}$$

$$= \frac{1}{2} \left(10 \operatorname{rect}\left(\frac{F-1100}{200}\right) + 10 \operatorname{rect}\left(\frac{F-500}{200}\right) \right) e^{-j10\pi F} e^{j8000\pi}$$

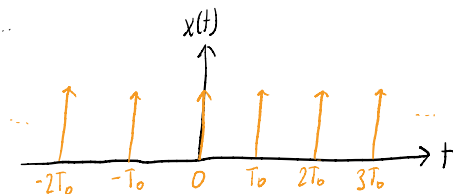
$$+ \frac{1}{2} \left(10 \operatorname{rect}\left(\frac{F+500}{200}\right) + 10 \operatorname{rect}\left(\frac{F+1100}{200}\right) \right) e^{-j10\pi F} e^{-j8000\pi}$$

$$\rightarrow Y(F) = \left(5 \operatorname{rect}\left(\frac{F-1100}{200}\right) + 5 \operatorname{rect}\left(\frac{F-500}{200}\right) + 5 \operatorname{rect}\left(\frac{F+500}{200}\right) + 5 \operatorname{rect}\left(\frac{F+1100}{200}\right) \right) e^{-j10\pi F}$$



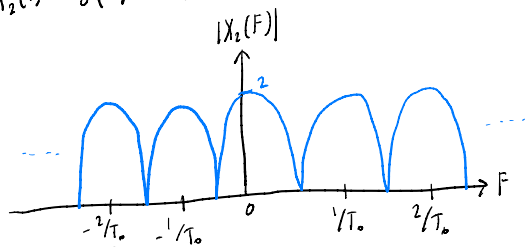
Just a few more relevant FT pairs/properties...

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$



Let's try just 2 to start:

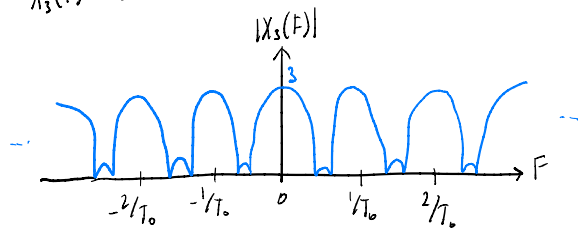
$$x_2(t) = \delta(t) + \delta(t - T_0) \xrightarrow{\mathcal{F}} X_2(F) = 1 + e^{-j2\pi FT_0}$$



- Sums to 2 when $e^{-j2\pi FT_0} = 1$
- Sums to 0 when $e^{-j2\pi FT_0} = -1$
- Periodic w/ period $1/T_0$!

How about 3?

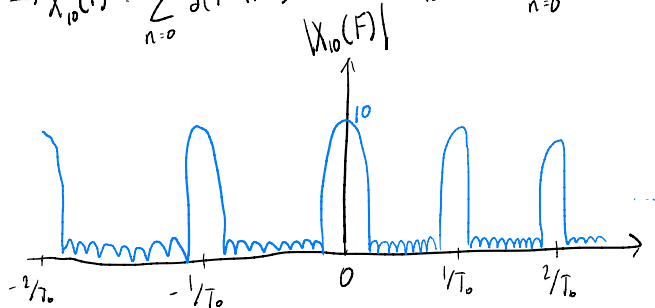
$$x_3(t) = \delta(t) + \delta(t - T_0) + \delta(t - 2T_0) \xrightarrow{\mathcal{F}} X_3(F) = 1 + e^{-j2\pi FT_0} + e^{-j4\pi FT_0}$$



- Peaks at 3, and wider "null area"
- Peaks get taller and narrower

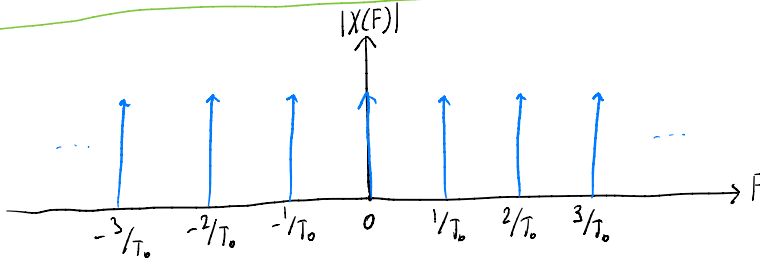
What about 10?

$$\rightarrow x_{10}(t) = \sum_{n=0}^9 \delta(t - nT_0) \xrightarrow{\mathcal{F}} X_{10}(F) = \sum_{n=0}^9 e^{-j2\pi n FT_0}$$



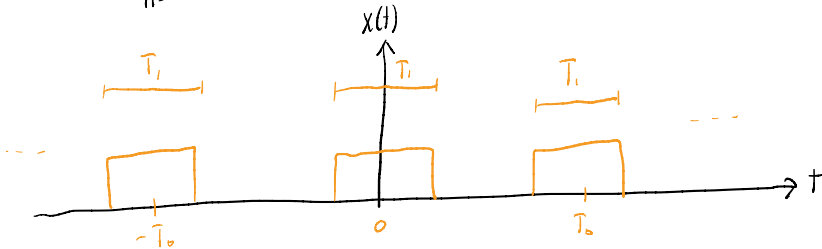
→ As we add more delta functions to the train, the periodic FT peaks get taller/narrower
 → Infinite delta functions means infinitely tall/narrow peaks → Delta functions in FT!

$$\rightarrow x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \xleftrightarrow{F} X(F) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta(F - n\frac{1}{T_0})$$



We can use impulse trains to make periodic functions:

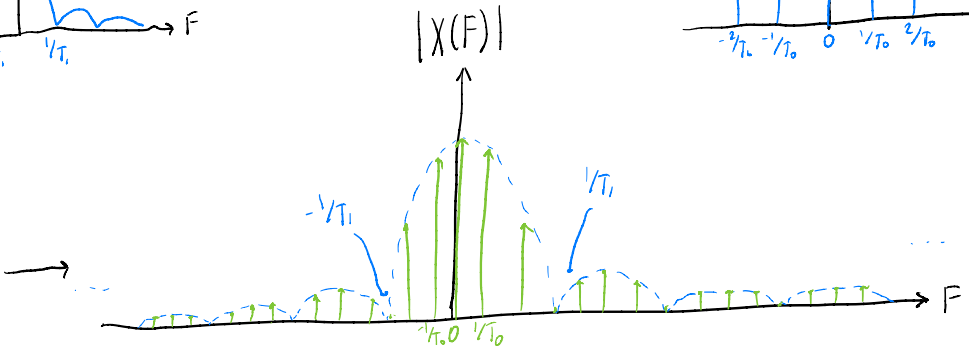
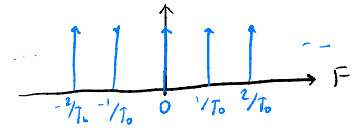
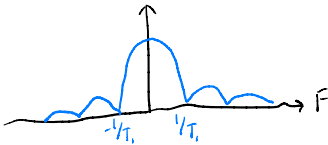
$$x(t) = \text{rect}\left(\frac{t}{T_1}\right) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{t - nT_0}{T_1}\right)$$



$$F\left\{\text{rect}\left(\frac{t}{T_1}\right)\right\} = T_1 \text{sinc}(T_1 F)$$

$F\{\text{convolution}\} = \text{multiplication}$

$$F\left\{\sum_{n=-\infty}^{\infty} \delta(t - nT_0)\right\} = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta(F - n\frac{1}{T_0})$$



- Periodic $x(t)$ gives a FT that is a "sampled" version of the FT of the aperiodic signal
- This sampled values correspond exactly to the Fourier Series coefficients a_k !

Main Idea: Aperiodic $x(t) \longleftrightarrow$ Continuous $X(F)$
 Periodic $x(t) \longleftrightarrow$ Discrete $X(F)$ (CT Fourier series)

Later on, we will observe a similar duality with DT $x[n]$.

Another relationship to Fourier Series: Symmetry

Remember the symmetries we have in FS for certain signals:

- $x(t)$ real $\rightarrow a_k = a_{-k}^*$ (conjugate symmetric)
- $x(t)$ real & even $\rightarrow a_k = a_{-k} = a_{-k}^*$ (real-valued & even)
- $x(t)$ real & odd $\rightarrow a_k = -a_{-k} = a_{-k}^*$ (imaginary-valued & odd)

We established above that a_k is essentially the FT for periodic FT, so we might expect analogous symmetry in $X(F)$... which, in fact, we observe:

Symmetry Properties of Fourier Transform:

- $x(t)$ real $\rightarrow X(F) = X^*(-F)$ $|X(F)| = |X(-F)|$ and $\angle X(F) = -\angle X(-F)$
 Conjugate Symmetric Magnitude Even Phase Odd
- $x(t)$ real & even $\rightarrow X(F) = X(-F)$ $|X(F)| = |X(-F)|$ and $\angle X(F) = 0$
 Real & Even Magnitude Even Purely Real
- $x(t)$ real & odd $\rightarrow X(F) = -X(-F)$ $|X(F)| = |X(-F)|$ and $\angle X(F) = \pm \frac{\pi}{2}$
 Imaginary & Odd Magnitude Even Purely Imaginary

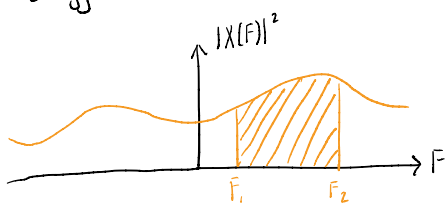
Parseval's Theorem

Signal energy can be calculated in either domain:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(F)|^2 dF = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Note: Parseval's Theorem in Fourier Series is for computing power, while Parseval's Theorem in the Fourier Transform is for computing energy.

We can use Parseval's Theorem to calculate total energy, or just how much energy is in a finite band of frequencies.



$$B \in [F_1, F_2] \rightarrow E_B = \int_{F_1}^{F_2} |X(F)|^2 dF$$

Example: $x(t) = A \text{sinc}(10t)$

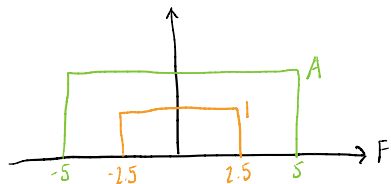
$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} A^2 \text{sinc}^2(10t) dt = ???$$

$$\rightarrow A \text{sinc}(10t) \xleftrightarrow{F} \frac{A}{10} \text{rect}\left(\frac{F}{10}\right)$$

$$\rightarrow E_x = \int_{-\infty}^{\infty} |X(F)|^2 dF = \int_{-5}^5 \left(\frac{A}{10}\right)^2 dF = \left. \frac{A^2}{100} F \right|_{-5}^5 = \boxed{\frac{A^2}{10}}$$

Now let $y(t) = h(t) * x(t)$ where $h(t) = 5 \text{sinc}(5t)$. Find E_y .

$$\begin{aligned} \rightarrow H(F) &= \text{rect}\left(\frac{F}{5}\right) & \rightarrow Y(F) &= H(F)X(F) = \frac{A}{10} \text{rect}\left(\frac{F}{5}\right) \text{rect}\left(\frac{F}{10}\right) \\ & & &= \frac{A}{10} \text{rect}\left(\frac{F}{5}\right) \end{aligned}$$



$$\begin{aligned} \rightarrow E_y &= \int_{-\infty}^{\infty} |Y(F)|^2 dF = \int_{-2.5}^{2.5} \left(\frac{A}{10}\right)^2 dF = \left. \frac{A^2}{100} F \right|_{-2.5}^{2.5} \\ &= \boxed{\frac{A^2}{20}} \end{aligned}$$

Signal Transmission through LTI Systems

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) \quad y(t) = x(t) * h(t)$$

From convolution property of FT:

$$Y(F) = X(F) H(F)$$

(this assumes system is BIBO stable - $H(F)$ must exist)

The operations of convolution in time and multiplication in frequency both allow us to calculate the same output, but each provides a different insight into how LTI systems work

Time Domain

Input signal $x(t)$ is a weighted sum (integral) of impulses

$$\rightarrow x(t) = \int_{-\infty}^{\infty} \underbrace{x(\tau)}_{\text{Signal value at } t=\tau} \underbrace{\delta(t-\tau)}_{\text{Impulse at } t=\tau \text{ weighted by } x(\tau)} d\tau$$

Signal value at $t=\tau$

Impulse at $t=\tau$ weighted by $x(\tau)$

Output signal $y(t)$ is a weighted sum of the responses to impulses

$$\rightarrow y(t) = \int_{-\infty}^{\infty} \underbrace{x(\tau)}_{\text{Signal weight at } t=\tau} \underbrace{h(t-\tau)}_{\text{Impulse response due to instantaneous component of } x(t) \text{ at } t=\tau} d\tau \quad \left. \vphantom{\int_{-\infty}^{\infty}} \right\} \text{Convolution Integral}$$

Signal weight at $t=\tau$

Impulse response due to instantaneous component of $x(t)$ at $t=\tau$

Frequency Domain

Input signal $x(t)$ is a weighted sum (integral) of sinusoids

$$\rightarrow x(t) = \int_{-\infty}^{\infty} \underbrace{X(F)}_{\text{Weight of frequency } F \text{ in signal}} \underbrace{e^{j2\pi Ft}}_{\text{Sinusoid of frequency } F \text{ weighted by } H(F)} dF \quad \left. \vphantom{\int_{-\infty}^{\infty}} \right\} \text{Inverse FT}$$

Weight of frequency F in signal

Sinusoid of frequency F weighted by $H(F)$

Output signal $y(t)$ is a weighted sum of the responses to sinusoids

$$\rightarrow y(t) = \int_{-\infty}^{\infty} \underbrace{X(F)}_{\text{Original frequency weight at } F} \underbrace{H(F)}_{\text{Modification/scaling factor of frequency } F \text{ due to the system}} e^{j2\pi Ft} dF$$

Original frequency weight at F

Modification/scaling factor of frequency F due to the system

$$Y(F) = X(F) H(F)$$

$$\rightarrow y(t) = \int_{-\infty}^{\infty} Y(F) e^{j2\pi Ft} dF$$

The response to an impulse is spread out over time by $h(t)$

The response to a sinusoid is described by a complex scalar value $H(F_0)$

$$Y(F) = X(F) H(F)$$

Frequency content of the input signal

Describes how each frequency is modified/distorted by the system

→ The "Frequency Response" of the system

- Some frequency components might be amplified, some might be attenuated, and some might stay the same

- Relative phase might change

→ The output waveform is changed from the input waveform as a function of frequency

Example:

$$x(t) = \cos(50\pi t) + \cos(200\pi t)$$

$$h(t) = 100 \operatorname{sinc}(100t) + (\operatorname{sinc}(100t) \cos(200\pi t)) * \delta(t - \frac{1}{8})$$

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) \quad y(t) = x(t) * h(t)$$

Find $X(F)$, $H(F)$, $Y(F)$, and $y(t)$.

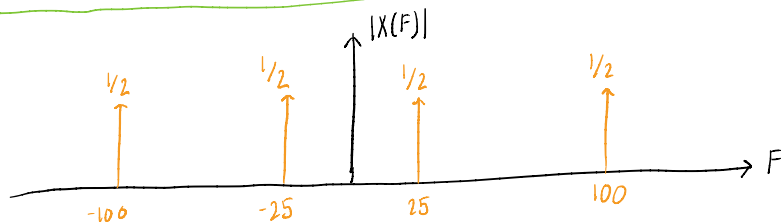
$$x(t) = \cos(50\pi t) + \cos(200\pi t)$$

$$F_0 = 25$$

$$F_0 = 100$$

$$\cos(2\pi F_0 t) \xleftrightarrow{F} \frac{1}{2} \delta(F - F_0) + \frac{1}{2} \delta(F + F_0)$$

$$\rightarrow X(F) = \frac{1}{2} \delta(F - 25) + \frac{1}{2} \delta(F + 25) + \frac{1}{2} \delta(F - 100) + \frac{1}{2} \delta(F + 100)$$



$$h(t) = 100 \operatorname{sinc}(100t) + (\operatorname{sinc}(100t) \cos(200\pi t)) * \delta(t - \frac{1}{8})$$

a.) Building blocks:

$$\begin{aligned}
 h_1(t) &= 100 \operatorname{sinc}(100t) \xleftrightarrow{F} \operatorname{rect}(F/100) &= H_1(F) \\
 h_2(t) &= \operatorname{sinc}(100t) \xleftrightarrow{F} \frac{1}{100} \operatorname{rect}(F/100) &= H_2(F) \\
 h_3(t) &= \cos(200\pi t) \xleftrightarrow{F} \frac{1}{2} \delta(F - 100) + \frac{1}{2} \delta(F + 100) &= H_3(F) \\
 h_4(t) &= \delta(t - \frac{1}{8}) \xleftrightarrow{F} e^{-j\pi/4 F} &= H_4(F)
 \end{aligned}$$

b.) No important building block 'solo' modifications

c.) Rewrite: $h(t) = h_1(t) + (h_2(t)h_3(t)) * h_4(t)$

Multiplication: $h_2(t)h_3(t) \xleftrightarrow{\mathcal{F}} H_2(F) * H_3(F)$

Convolution: $(h_2(t)h_3(t)) \xleftrightarrow{\mathcal{F}} (H_2(F) * H_3(F)) H_4(F)$

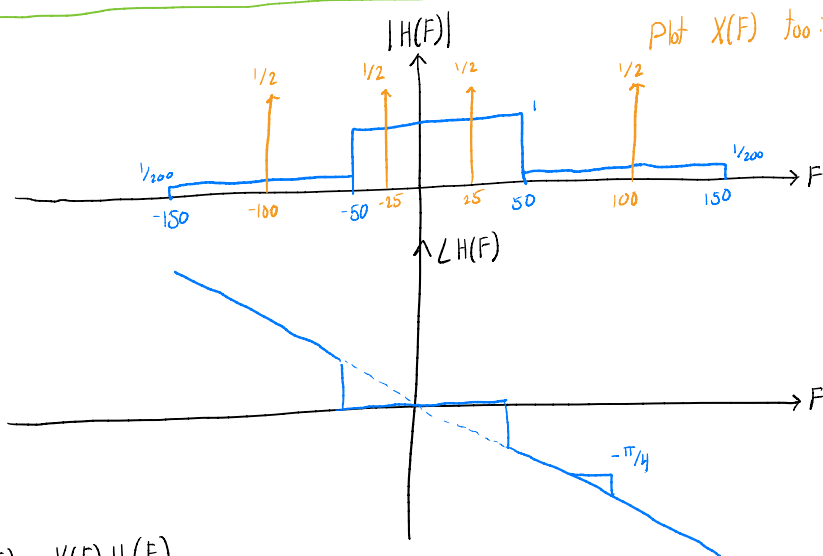
Linearity: $h(t) = h_1(t) + (h_2(t)h_3(t)) * h_4(t) \xleftrightarrow{\mathcal{F}} H_1(F) + (H_2(F) * H_3(F)) H_4(F) = H(F)$

$$H_2(F) * H_3(F) = \frac{1}{100} \text{rect}\left(\frac{F}{100}\right) * \left(\frac{1}{2} \delta(F-100) + \frac{1}{2} \delta(F+100)\right) = \frac{1}{200} \text{rect}\left(\frac{F-100}{100}\right) + \frac{1}{200} \text{rect}\left(\frac{F+100}{100}\right)$$

Substitute in:

$$\rightarrow H(F) = \text{rect}\left(\frac{F}{100}\right) + \left(\frac{1}{200} \text{rect}\left(\frac{F-100}{100}\right) + \frac{1}{200} \text{rect}\left(\frac{F+100}{100}\right)\right) e^{-j\frac{\pi}{4}F}$$

d.)



Finally: $Y(F) = X(F)H(F)$

$$\begin{aligned} &= \frac{1}{2} H(25) \delta(F-25) + \frac{1}{2} H(-25) \delta(F+25) + \frac{1}{2} H(100) \delta(F-100) + \frac{1}{2} H(-100) \delta(F+100) \\ &= \frac{1}{2} \delta(F-25) + \frac{1}{2} \delta(F+25) + \frac{1}{400} \delta(F-100) e^{-j25\pi} + \frac{1}{400} \delta(F+100) e^{j25\pi} \end{aligned}$$

$$= \left(\frac{1}{2} \delta(F-25) + \frac{1}{2} \delta(F+25)\right) - \left(\frac{1}{400} \delta(F-100) + \frac{1}{400} \delta(F+100)\right)$$

$$\rightarrow y(t) = \cos(50\pi t) - \frac{1}{200} \cos(200\pi t)$$

Same freq. sinusoids, but one is unchanged and the other is significantly attenuated and phase-shifted by 180°