# Complex Exponentials:

$e^{j\omega_0 t} \longrightarrow \omega_0$ is radial frequency in radians/s

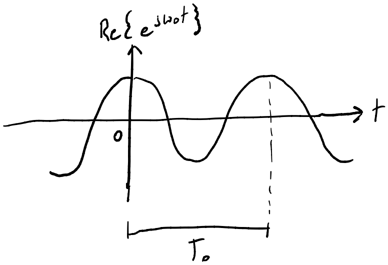
Can also write as $e^{j2\pi F_0 t} \rightarrow F_0$ is cyclic frequency in cycles/s or Hertz (Hz)

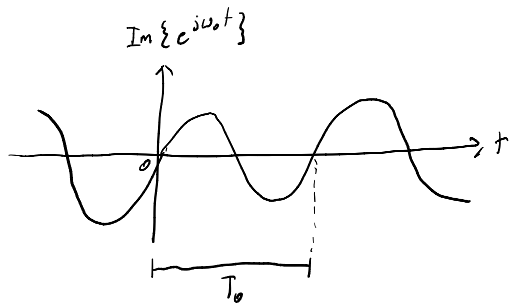
$2\pi$ radians per cycle $\rightarrow$ $\boxed{F_0 = \frac{\omega_0}{2\pi}}$ and $\boxed{\omega_0 = 2\pi F_0}$

Period: $\boxed{T_0 = \frac{1}{F_0} = \frac{2\pi}{\omega_0}}$ $\rightarrow$ The number of seconds in one cycle

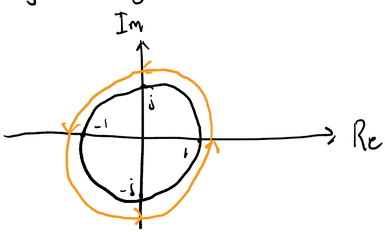
## Euler's Thm.: $e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$

One way: Real and imaginary




Other way: rotating phasor in complex plane



Complex exponentials have very nice properties with LTI systems!

Exponentials $e^{j\omega_0 t}$ (or more generally, $e^{st}$ w/ $s = \sigma + j\omega$) are eigenfunctions of LTI systems

Recall from linear algebra: sometimes have matrices/vectors such that

$$A\bar{x} = \lambda \bar{x}$$

Matrix, Vector, Scalar, Same vector

→ The matrix scales the vector but doesn't change its direction

→ $\bar{x}$ is an eigenvector of $A$; $\lambda$ is associated eigenvalue

Similarly for $e^{j\omega_0 t}$ as an eigenfunction:

$$e^{j\omega_0 t} \rightarrow \boxed{LTI} \rightarrow \lambda e^{j\omega_0 t}$$

complex scalar (amplitude & phase)

LTI systems scale complex exponentials (i.e., change amplitude and phase) but don't change the functional form → $\omega_0$ doesn't change!

Proof using convolution:

Convolution gives system output given system input and impulse response

$$y(t) = x(t) * h(t-\tau) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Let $x(t) = e^{j\omega_0 t}$

$$\rightarrow y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega_0 (t-\tau)}\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega_0 t} e^{-j\omega_0 \tau}\, d\tau = \boxed{\underbrace{e^{j\omega_0 t}}_{\text{Same exponential}} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega_0 \tau}\, d\tau}_{\text{A constant } H(\omega_0)}}$$

(More later...)

Remember linearity: $a_1 e^{j\omega_1 t} + a_2 e^{j\omega_2 t} + \dots \rightarrow \boxed{LTI} \rightarrow \lambda_1 a_1 e^{j\omega_1 t} + \lambda_2 a_2 e^{j\omega_2 t} + \dots$

So, if we have a more complex $x(t)$, it would be great if we could express it as a summation of exponentials since that will make it easier to determine the output of some LTI system!

# Fourier Series for Continuous - Time Periodic Signals

Periodic signal: $x(t) = x(t-T)$ for some $T$ for all $t$

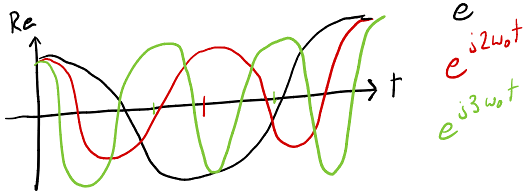
- The smallest value of $T$ for which this holds: $T_0 \triangleq$ fundamental period
- $F_0 = 1/T_0 \triangleq$ fundamental frequency
- Similarly refer to $\omega_0 = 2\pi F_0 = \frac{2\pi}{T_0}$ as fundamental frequency

Signal $x(t) = e^{j\omega_0 t} \rightarrow$ periodic

Harmonic signals: $\phi_k(t) = e^{jk\omega_0 t}$ $\quad$ $k$ is an integer

The frequency of the $k^{th}$ harmonic is $k$ times the fundamental frequency $\omega_0$

- Sometimes call the exponential at the fundamental frequency the "1st harmonic"
- Similarly have the 2nd, 3rd, etc. harmonics




It turns out that any* periodic function can be written as a big sum (series) of its harmonics!

* → must meet some regularity conditions

Let $x(t)$ be a periodic signal with fund. freq. $\omega_0$

We can express $x(t)$ as:

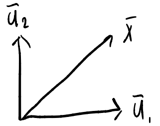
$$x(t) = \sum_{k=-\infty}^{\infty} \underbrace{a_k}_{\text{complex constant}} \underbrace{e^{jk\omega_0 t}}_{k^{th} \text{ harmonic}}$$

The constant $a_k$ tells us "how much" of each harmonic $e^{jk\omega_0 t}$ is present in $x(t)$

This is called the Fourier series representation of $x(t)$

How do we determine $a_k$ (i.e., "how much" of the $k^{th}$ harmonic is in $x(t)$)?

Remember vector dot products:




How to find $\bar{x}$ in terms of $\bar{u}_1$ and $\bar{u}_2$?

$\bar{x} = \underline{a}\bar{u}_1 + \underline{b}\bar{u}_2$

$\langle \cdot \rangle$ = dot/inner product

$a$ = projection of $\bar{x}$ onto $\bar{u}_1$ $= \langle \bar{x}, \bar{u}_1 \rangle$

How much of $\bar{u}_1$ is in $\bar{x}$?

$b$ = projection of $\bar{x}$ onto $\bar{u}_2$ $= \langle \bar{x}, \bar{u}_2 \rangle$

How much of $\bar{u}_2$ is in $\bar{x}$?

Ex. $\bar{a} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\quad \bar{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\langle \bar{a}, \bar{b} \rangle = \underset{a_0}{1} \cdot \underset{b_0}{1} + \underset{a_1}{3} \cdot \underset{b_1}{0} = \underline{1}$$

→ Arbitrary N-length dot product: $\langle \bar{a}, \bar{b} \rangle = \sum_{k=0}^{N-1} a_k b_k$

For complex vectors: $\langle \bar{a}, \bar{b} \rangle = \sum_{k=0}^{N-1} a_k \underset{\text{conjugate}}{b_k^*}$ → allows $\langle \bar{a}, \bar{a} \rangle = \sum_{k=0}^{N-1} |a_k|^2$

Generalizes to infinite-length vectors: $\langle \bar{a}, \bar{b} \rangle = \sum_{k=-\infty}^{\infty} a_k b_k^*$

The vectors above are discrete, but continuous functions can also be treated as vectors: $\underline{\langle a(t), b(t) \rangle = \int_{-\infty}^{\infty} a(t) b^*(t)\, dt}$ dot product for CT functions

Just like for the vectors in geometric space, this dot product "projects" $a(t)$ onto $b(t)$ to tell us "how much of $b(t)$ is in $a(t)$"?

Back to Fourier series: $a_k$ tells us "how much" of $e^{jk\omega_0 t}$ is in $x(t)$

→ We find $a_k$ by projection $x(t)$ onto $e^{jk\omega_0 t}$!

Recall: $(e^{j\theta})^* = e^{-j\theta}$

$$a_k = \langle x(t), e^{jk\omega_0 t} \rangle$$

$$= \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt$$

Period of $T_0$ Period of $T_0/k$

→ Integral repeats every $T_0$ seconds
Full integral yields infinite results,* so actually only want to integrate over one period and average

*A little hand-wavy. There is a better proof in the book, but we're engineers, so proofs will not be on the exams...

$$\rightarrow a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

can integrate over any bounds totaling one period

## Fourier Series Equations for Periodic Signals:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$ → synthesis equation

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$ → analysis equation

$a_k$ called "Fourier series coefficients" or "spectral coefficients" of $x(t)$

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t) dt$$ → $a_0$ is the average value of one period of $x(t)$

"the DC component"

# Fourier Series for real signals:

1. $x(t) = x^*(t) \rightarrow \quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$

$$\rightarrow \boxed{a_k^* = a_{-k}}$$

Proofs are on pgs. 188-189 of book

2. Alternate FS representations:

a.) $\boxed{\text{Let } a_k = A_k e^{j\theta_k} \longrightarrow x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)}$

Polar form

b.) $\boxed{\text{Let } a_k = B_k + jC_k \longrightarrow x(t) = a_0 + 2\sum_{k=1}^{\infty} \left[B_k \cos(k\omega_0 t) - C_k \sin(k\omega_0 t)\right]}$

Rectangular form

$$\rightarrow \boxed{\begin{aligned} a_0 &= \frac{1}{T_0}\int_{T_0} x(t)\,dt \quad && \text{(DC value)} \\ B_k &= \frac{1}{T_0}\int_{T_0} x(t)\cos(k\omega_0 t)\,dt \quad && k \neq 0 \\ C_k &= -\frac{1}{T_0}\int_{T_0} x(t)\sin(k\omega_0 t)\,dt \quad && k \neq 0 \end{aligned}}$$

* Note: these equations are <u>different</u> from the ones in the Nilsson and Riedel Circuits textbook!

# Convergence of FS

3 regularity conditions (Dirichlet Conditions):

1.) $x(t)$ must be absolutely integrable

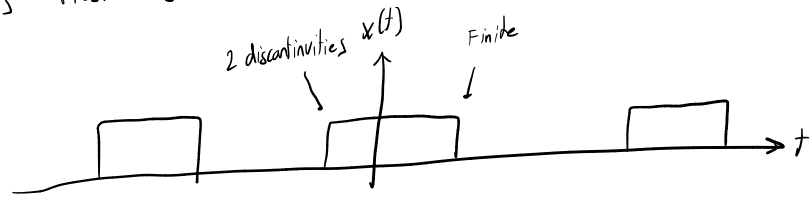
$$\int_{T_0} |x(t)|\,dt < \infty \qquad \text{(i.e., finite } a_k\text{)}$$

2.) $x(t)$ must have a finite number of minima & maxima in one period

ex. $x(t) = \cos\left(\frac{2\pi}{t}\right) \quad 0 \le t < 1$

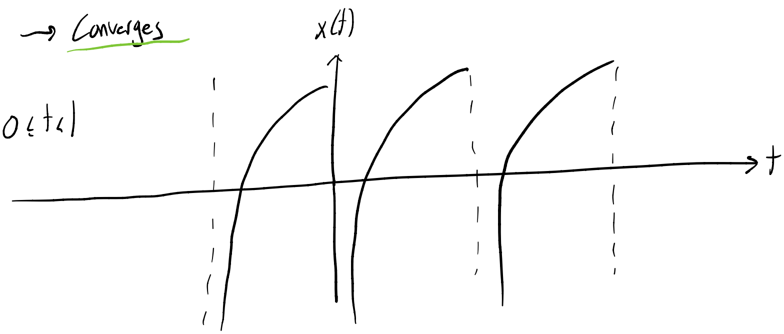
3.) $x(t)$ must have a finite number of discontinuities in one period and all discontinuities must be finite

Ex.



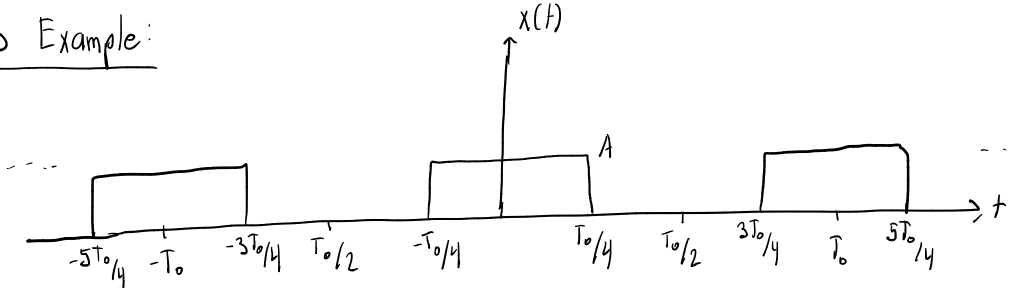

→ Converges

$x(t) = \log(x) \quad 0 \le t < 1$




→ Does not converge

# FS Example:




Find $a_k$.

$T_0 \rightarrow$ fundamental period

$\omega_0 \rightarrow \frac{2\pi}{T_0}$ and $F_0 = \frac{1}{T_0}$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,dt = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} A\,dt = \frac{A}{T}\,t\Big|_{-T/4}^{T/4} = \frac{A}{2}$$

$$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} Ae^{-jk\omega_0 t}dt$$

$$= \frac{A}{T_0}\left[\frac{1}{-jk\omega_0}e^{-jk\omega_0 t}\right]_{-T_0/4}^{T_0/4} = \frac{A}{-jk\omega_0 T_0}\left(e^{-jk\omega_0 T_0/4} - e^{jk\omega_0 T_0/4}\right)$$

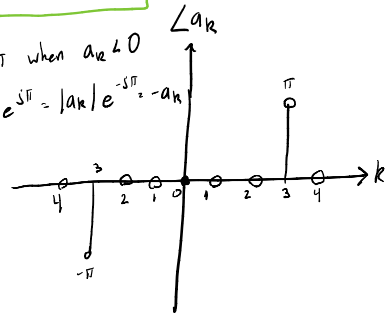
$$= \frac{A}{jk\omega_0 T_0}\underbrace{\left(e^{jk\omega_0 T_0/4} - e^{-jk\omega_0 T_0/4}\right)}_{=2j\sin(k\omega_0 T_0/4)} = \frac{A}{jk\omega_0 T_0}\,2j\sin\left(k\omega_0\frac{T_0}{4}\right)$$

$$= \frac{2A}{k\omega_0 T_0}\sin\left(k\omega_0\frac{T_0}{4}\right) = \frac{2A}{k\frac{2\pi}{T_0}T_0}\sin\left(k\frac{2\pi}{T_0}\frac{T_0}{4}\right) = \boxed{\frac{A}{k\pi}\sin\left(k\frac{\pi}{2}\right) = a_k}$$

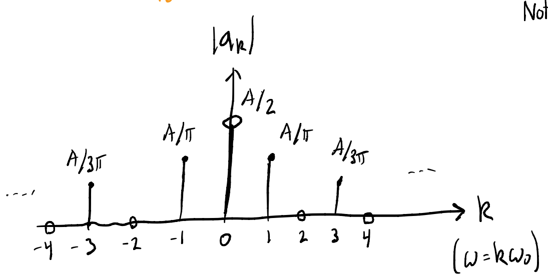
$\omega_0 = \frac{2\pi}{T_0}$

Phase $= \pm\pi$ when $a_k < 0$

Note: $|a_k|e^{j\pi} = |a_k|e^{-j\pi} = -a_k$

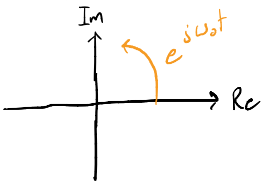



Stem plots:

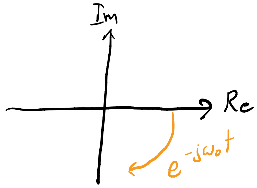


A note on negative frequencies:

- Imaginary numbers are mathematical constructs, not physical ones




Positive Frequency




Negative Frequency

But if we have $a_k = a_{-k}^*$, then $a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t} = 2a_k \cos(k\omega_0 t)$

$\rightarrow$ The imaginary parts cancel and we get a real cosine

---

## Properties of Fourier Series:

1.) Linearity: Suppose $x_1(t)$ and $x_2(t)$ both periodic w/ same period

$x_1(t) \overset{FS}{\longleftrightarrow} a_k \quad x_2(t) \overset{FS}{\longleftrightarrow} b_k \quad \Longrightarrow \quad x_1(t) + x_2(t) \overset{FS}{\longleftrightarrow} a_k + b_k$

2.) Time-shifting: If $x(t) \overset{FS}{\longleftrightarrow} a_k$ then $x(t-t_0) \overset{FS}{\longleftrightarrow} a_k e^{-jk\omega_0 t_0}$

3.) Time-reversal: If $x(t) \overset{FS}{\longleftrightarrow} a_k$ then $x(-t) \overset{FS}{\longleftrightarrow} a_{-k}$

a.) Symmetry: $x(t)$ even $(x(t) = x(-t)) \rightarrow a_k = a_{-k}$

$x(t)$ odd $(x(t) = -x(-t)) \rightarrow a_k = -a_{-k}$ and $a_0 = 0$

b.) Recall that if $x(t)$ is real, then $a_k = a_{-k}^*$

$x(t)$ real & even $\rightarrow a_k = a_{-k}^* = a_{-k} \rightarrow a_k$ must be real and even

$x(t)$ real & odd $\rightarrow a_k = a_{-k}^* = -a_{-k} \rightarrow a_k$ must be imaginary and odd

4.) Time-scaling: $x(t) \overset{FS}{\longleftrightarrow} a_k$ then $x(\alpha t) \overset{FS}{\longleftrightarrow} a_k$

$\rightarrow$ Same coefficients $a_k$, but new fund. freq. $\alpha\omega_0$

5.) Multiplication: If $x(t) \overset{FS}{\longleftrightarrow} a_k$ and $y(t) \overset{FS}{\longleftrightarrow} b_k$

Then $\underline{y(t)x(t) \overset{FS}{\longleftrightarrow} \sum_{m=-\infty}^{\infty} a_m b_{k-m}}$ (discrete convolution... more on this later)

6.) Conjugation: If $x(t) \overset{FS}{\longleftrightarrow} a_k$ then $\underline{x^*(t) \overset{FS}{\longleftrightarrow} a^*_{-k}}$

→ If $x(t)$ is real, then $x(t) = x^*(t)$ and $\underline{a_k = a^*_{-k}}$

→ $\underline{|a_k| = |a_{-k}|}$

$\underline{\angle a_k = -\angle a_{-k}}$

7.) Parseval's Relation: $\underline{\frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2}$ → Average power is preserved in both domains

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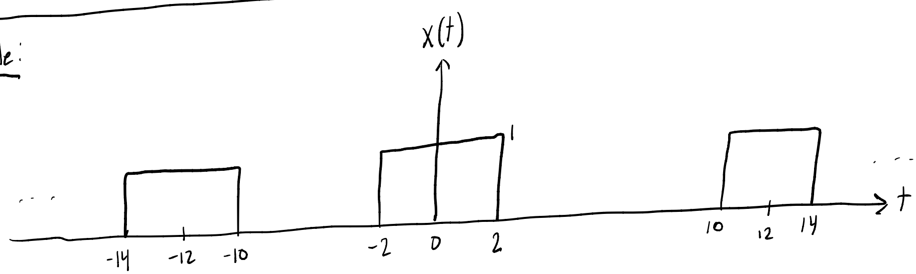
FS Example:

a.)




Find $a_k$ (can skip $a_0$).

First: $T_0 = 12$, $F_0 = \frac{1}{12}$, $\omega_0 = \frac{2\pi}{12} = \frac{\pi}{6}$

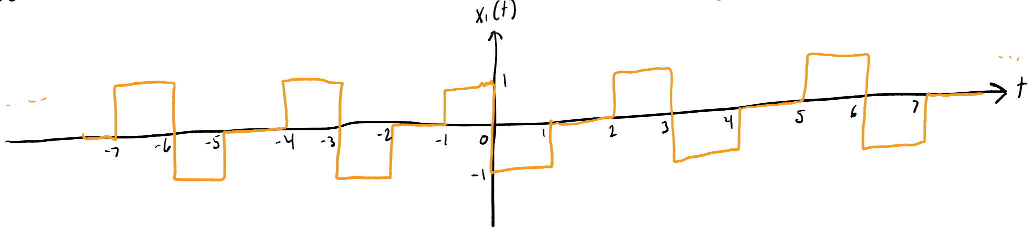
$$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T_0}\int_{-6}^{6} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T_0}\int_{-2}^{2} e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T_0}\left[\frac{1}{-jk\omega_0}e^{-jk\omega_0 t}\right]_{-2}^{2} = \frac{1}{-jk\omega_0 T_0}\left(e^{-j2k\omega_0} - e^{j2k\omega_0}\right) = \frac{1}{k\omega_0 T_0}\frac{1}{j}\underbrace{\left(e^{j2k\omega_0} - e^{-j2k\omega_0}\right)}_{= 2\sin(2k\omega_0)}$$

$$= \frac{2}{k\omega_0 T_0}\sin(2k\omega_0) = \boxed{\frac{\sin(k\frac{\pi}{3})}{k\pi}}$$

$\omega_0 T_0 = 2\pi$ $\qquad \omega_0 = \frac{\pi}{6}$

b.) Determine new $a_k$ of $x_1(t)$ from previous answer using pairs & properties.



New $T_0 \rightarrow T_0 = 3, \quad F_0 = \frac{1}{3}, \quad \omega_0 = \frac{2\pi}{3}$

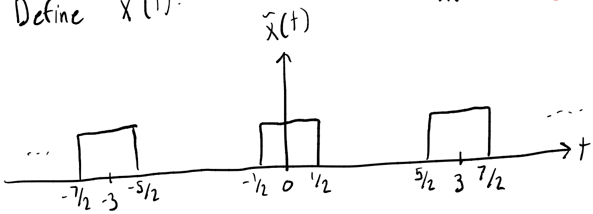
An observation: $x_1(t)$ is a linear combination of two time-scaled versions of $x(t)$, both of which are shifted in time and one of which is negated

Define $\tilde{x}(t)$:

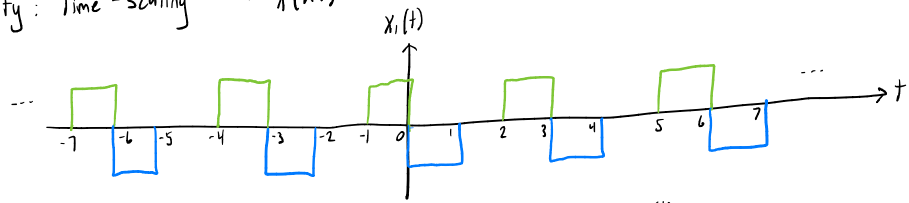
$\tilde{x}(t)$ is time-scaled copy of $x(t)$

$\rightarrow$ 4x "faster"

$\rightarrow \tilde{x}(t) = x(4t)$



FS property: Time-scaling $\rightarrow x(\alpha t) \overset{FS}{\longleftrightarrow} a_k$ i.e., $a_k$ does not change, but $T_0/\omega_0$ do!



$x_1(t)$ is a linear combination of time-shifted copies of $\tilde{x}(t)$:

$$x_1(t) = \tilde{x}\left(t + \tfrac{1}{2}\right) - \tilde{x}\left(t - \tfrac{1}{2}\right)$$

FS Property: Time-shifting $\rightarrow x(t - t_0) \overset{FS}{\longleftrightarrow} a_k e^{-jk\omega_0 t_0}$

$\rightarrow \tilde{x}\left(t + \frac{1}{2}\right) \overset{FS}{\longleftrightarrow} a_k e^{j\frac{1}{2}k\omega_0}$ $\quad \rightarrow \tilde{x}\left(t - \frac{1}{2}\right) \overset{FS}{\longleftrightarrow} a_k e^{-j\frac{1}{2}k\omega_0}$

FS Property: Linearity $\rightarrow x(t) + y(t) \overset{FS}{\longleftrightarrow} a_k + b_k$

$$\rightarrow x_1(t) \overset{FS}{\longleftrightarrow} a_k e^{j\frac{1}{2}k\omega_0} - a_k e^{-j\frac{1}{2}k\omega_0} = \underbrace{a_k}_{= \frac{\sin(k\pi/3)}{k\pi}} \underbrace{\left(e^{jk\frac{\pi}{3}} - e^{-jk\frac{\pi}{3}}\right)}_{= 2j\sin\left(k\frac{\pi}{3}\right)} = \boxed{\frac{2j}{k\pi}\sin^2\left(k\frac{\pi}{3}\right)}$$

# The Frequency Domain:

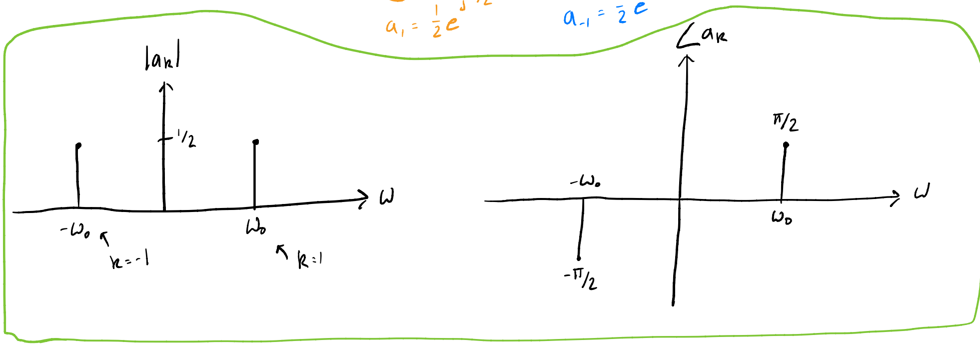
Time-domain: what does $x(t)$ look like as a function of time?

e.g., $x(t) = \cos(\omega_0 t + \pi/2)$

Frequency-domain: What does a signal look like as a combination of different frequency sinusoids? $\downarrow$ $e^{j\omega_0 t}$

e.g., $$x(t) = \cos(\omega_0 t + \pi/2) = \frac{1}{2} e^{j(\omega_0 t + \pi/2)} + \frac{1}{2} e^{-j(\omega_0 t + \pi/2)}$$

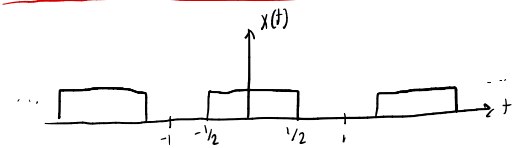
$$= \underbrace{\frac{1}{2} e^{j\pi/2}}_{a_1 = \frac{1}{2} e^{j\pi/2}} e^{j\omega_0 t} + \underbrace{\frac{1}{2} e^{-j\pi/2}}_{a_{-1} = \frac{1}{2} e^{-j\pi/2}} e^{-j\omega_0 t}$$




Fourier series (and other transforms later) let us look at signals in terms of "how much" of each $e^{j\omega t}$ we have in $x(t)$ as a function of $\omega$ (or $F$)

$\rightarrow$ Independent axis is frequency, not time $\rightarrow$ Frequency-domain!

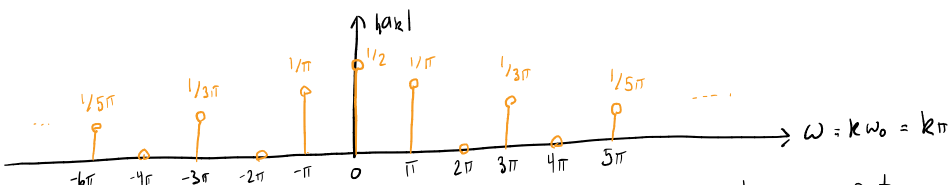
Example:




$T_0 = 2, \quad \omega_0 = \pi$

$a_0 = \frac{1}{2}$

We worked this example earlier... $\rightarrow$ $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ and $a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt = \frac{\sin(k\frac{\pi}{2})}{k\pi}$




$$x(t) = \frac{1}{5\pi} e^{-j5\pi t} - \frac{1}{3\pi} e^{-j3\pi t} + \frac{1}{\pi} e^{-j\pi t} + \frac{1}{2} e^{j0t} + \frac{1}{\pi} e^{j\pi t} - \frac{1}{3\pi} e^{j3\pi t} + \frac{1}{5\pi} e^{j5\pi t} + \ldots$$

Note: I did not draw the $\angle a_k$ plot

# LTI System Function and Filters

Recall: $e^{j\omega_0 t} \rightarrow \boxed{LTI} \rightarrow y(t) = e^{j\omega_0 t} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-j\omega_0 \tau} d\tau}_{H(\omega_0)} = \underbrace{H(\omega_0)}_{\text{System Function}} e^{j\omega_0 t}$

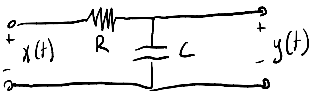
We will learn more about computing the system function later.
For now, notice how we can multiply the system function $H(\omega_0)$ with $e^{j\omega_0 t}$ to determine $y(t)$

$$a_1 e^{j\omega_1 t} + a_2 e^{j\omega_2 t} + \ldots \rightarrow \boxed{LTI} \rightarrow y(t) = a_1 H(\omega_1) e^{j\omega_1 t} + a_2 H(\omega_2) e^{j\omega_2 t} + \ldots$$

→ We can multiply $H(\omega)$ with the frequency-domain of $x(t)$ to get the frequency-domain of $y(t)$!

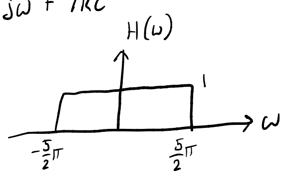
All (stable) LTI systems have an $H(\omega)$

Ex.

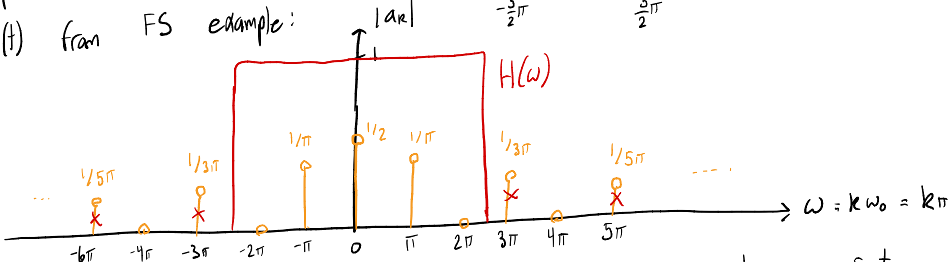



$$\rightarrow H(\omega) = \frac{1/RC}{j\omega + 1/RC}$$

Another example: $H(\omega) = \text{rect}\left(\frac{\omega}{5\pi}\right) \rightarrow$



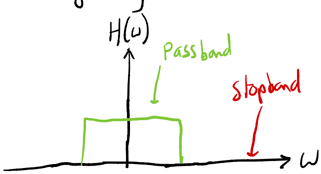

Apply to $x(t)$ from FS example:



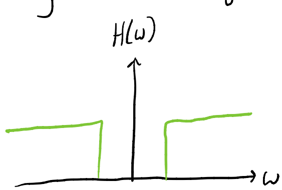

$$x(t) = \frac{1}{5\pi} e^{-j5\pi t} - \frac{1}{3\pi} e^{-j3\pi t} + \frac{1}{\pi} e^{-j\pi t} + \frac{1}{2} e^{j0t} + \frac{1}{\pi} e^{j\pi t} - \frac{1}{3\pi} e^{j3\pi t} + \frac{1}{5\pi} e^{j5\pi t} + \ldots$$

$$\rightarrow \boxed{y(t) = \frac{1}{2} + \frac{1}{\pi} e^{-j\pi t} + \frac{1}{2} e^{j\pi t}}$$

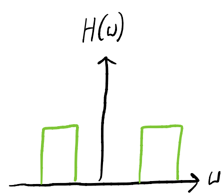
The LTI system described above is an ideal low-pass filter (LPF).
We generally have the following ideal filter types:



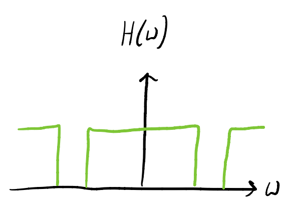

Low-pass filter (LPF)




High-pass filter (HPF)




Band-pass filter (BPF)




Band-stop filter (BSF)