

Chapter 4: Propagation of States and Covariances

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In recursive least squares (RLS) estimation, the goal is to estimate a constant $\bar{x}$ from a time series of measurements $\bar{z}_k$

For many estimation problems, we will be interested in estimating a vector $\bar{x}_k$ that changes with time

→ Often, $\bar{x}_k$ describes the state of a system → State Estimation

In RLS, we keep track of an estimate $\hat{x}_k$ of $\bar{x}$ as well as the covariance P_k of our estimate, with P_k only being changed w/ new measurements $\bar{z}_k$

→ Since $\bar{x}_k$ now changes (potentially including noise), P_k will also be modified by state transitions

DT Systems:

Suppose we have the following DT system dynamic:

$$\bar{x}_k = F_{k-1} \bar{x}_{k-1} + G_{k-1} \bar{u}_{k-1} + \bar{w}_{k-1}$$

where $\bar{w}_{k-1}$ is zero-mean, white, and Gaussian distributed w/ covariance Q_k

→ $\bar{w}_{k-1} \sim N(0, Q_k)$

→ $\bar{w}_{k-1}$ is called process noise and represents modeling errors in our state update or errors in our knowledge of $\bar{u}_{k-1}$

→ Q_k is called process covariance

Because of process noise, $\bar{x}_k$ is an RV

- Suppose $E\{\bar{x}_k\} = \bar{\mu}_k$ and $\text{cov}\{\bar{x}_k\} = P_k$

- We want to see how the mean and covariance of $\bar{x}_k$ change w/ time

Going back to the state equation $\rightarrow$ How does $\bar{M}_k$ change from $k-1$ to k ? (2)

$$E\{\bar{X}_k\} = \bar{M}_k = E\{F_{k-1}\bar{X}_{k-1} + G_{k-1}\bar{U}_{k-1} + \bar{W}_k\}$$

$$= \boxed{F_{k-1}\bar{M}_{k-1} + G_{k-1}\bar{U}_{k-1}}$$

$\rightarrow$ Because $\bar{W}_k$ is zero-mean, the mean value is propagated through the system update exactly as if it were the true state
 $\rightarrow$ We will use this property in the Kalman filter for prediction

How does covariance P_k change from $k-1$ to k ?

$$\begin{aligned} \rightarrow P_k &= E\{(\bar{X}_k - \bar{M}_k)(\bar{X}_k - \bar{M}_k)^T\} \\ &= E\{(F_{k-1}\bar{X}_{k-1} + G_{k-1}\bar{U}_{k-1} + \bar{W}_{k-1} - \bar{M}_k)(F_{k-1}\bar{X}_{k-1} + G_{k-1}\bar{U}_{k-1} + \bar{W}_{k-1} - \bar{M}_k)^T\} \\ &= E\{(F_{k-1}(\bar{X}_{k-1} - \bar{M}_{k-1}) + \bar{W}_{k-1})(F_{k-1}(\bar{X}_{k-1} - \bar{M}_{k-1}) + \bar{W}_{k-1})^T\} \\ &= E\{F_{k-1}(\bar{X}_{k-1} - \bar{M}_{k-1})(\bar{X}_{k-1} - \bar{M}_{k-1})^T F_{k-1}^T\} + E\{\bar{W}_{k-1}\bar{W}_{k-1}^T\} \\ &\quad + E\{F_{k-1}(\bar{X}_{k-1} - \bar{M}_{k-1})\bar{W}_{k-1}^T\} + E\{\bar{W}_{k-1}(\bar{X}_{k-1} - \bar{M}_{k-1})F_{k-1}^T\} \\ &= F_{k-1} E\{(\bar{X}_{k-1} - \bar{M}_{k-1})(\bar{X}_{k-1} - \bar{M}_{k-1})^T\} F_{k-1}^T + E\{\bar{W}_{k-1}\bar{W}_{k-1}^T\} \\ &\quad + F_{k-1} E\{(\bar{X}_{k-1} - \bar{M}_{k-1})\bar{W}_{k-1}^T\} + E\{\bar{W}_{k-1}(\bar{X}_{k-1} - \bar{M}_{k-1})\} F_{k-1}^T \end{aligned}$$

$\rightarrow \bar{W}_{k-1}$ is zero-mean and uncorrelated w/ $\bar{X}_{k-1}$, so these terms = 0

$$= \boxed{F_{k-1} P_{k-1} F_{k-1}^T + Q_{k-1}} = P_k$$

$\rightarrow$ Covariance is modified by F_{k-1} and then increased by Q_{k-1}

- Recall that if a random vector X is Gaussian, then any linear modification $Y = AX + \bar{b}$ is also Gaussian
- The sum of any Gaussian RVs is also Gaussian

→ If $\bar{X}_0$ is Gaussian, and so are all $\bar{w}_k$, then $\bar{X}_k$ will also always be Gaussian → $\bar{X}_k \sim N(\bar{\mu}_k, P_k)$ → Gaussians are perfectly characterized by mean & covariance, so in this sense we have full knowledge of the PDF of $\bar{X}_k$!

Suppose F_k and Q_k are time-invariant ($F_k = F$, $Q_k = Q$)

If F is stable (i.e., all eigenvalues $|\lambda_i| < 1$), then as $k \rightarrow \infty$ the covariance P_k will approach a steady-state value:

$$P = \sum_{i=0}^{\infty} F^i Q (F^T)^i$$

Sampled-Data Systems:

DT systems typically approximations of CT systems

→ How can we best approximate covariance of CT state in DT?

Recall: $\dot{\bar{X}} = A\bar{X} + B\bar{u} + \bar{w}$ → The noise term is new

$$\text{Solution: } \bar{X}(t_k) = e^{A(t_k - t_{k-1})} \bar{X}(t_{k-1}) + \int_{t_{k-1}}^{t_k} e^{A(t_k - \tau)} (B(\tau) \bar{u}(\tau) + \bar{w}(\tau)) d\tau$$

Let $\Delta t = t_k - t_{k-1}$, $\bar{X}_k = \bar{X}(t_k)$, and $\bar{u}_k = \bar{u}(t_k)$ (assuming $\bar{u}(t)$ is piecewise constant)

$$\rightarrow \bar{X}_k = e^{A\Delta t} \bar{X}_{k-1} + \int_{t_{k-1}}^{t_k} e^{A(t_k - \tau)} B(\tau) d\tau \bar{u}_{k-1}$$

$$F_k = e^{A\Delta t}$$

$$G_k = \int_{t_k}^{t_{k+1}} e^{A(t_{k+1} - \tau)} B(\tau) d\tau$$

$$\rightarrow \bar{X}_k = F_{k-1} \bar{X}_{k-1} + G_{k-1} \bar{u}_{k-1} + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} \bar{w}(\tau) d\tau$$

Assume $\bar{w}(t)$ is zero-mean, white, and Gaussian

Mean: $\bar{u}_k = E\{\bar{X}_k\} = F_{k-1} \bar{u}_{k-1} + G_{k-1} \bar{u}_{k-1} \rightarrow \text{Same as DT system}$

Covariance:
$$P_k = E\{(\bar{X}_k - \bar{u}_k)(\bar{X}_k - \bar{u}_k)^T\}$$

$$= E\left\{\left(F_{k-1} \bar{X}_{k-1} + G_{k-1} \bar{u}_{k-1} + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} \bar{w}(\tau) d\tau - \bar{X}_k\right) \left(\dots\right)^T\right\}$$

Same expression

$$= F_{k-1} P_{k-1} F_{k-1}^T + E\left\{\left(\int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} \bar{w}(\tau) d\tau\right) \left(\int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} \bar{w}(\tau) d\tau\right)^T\right\}$$

$$= F_{k-1} P_{k-1} F_{k-1}^T + \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} E\{\bar{w}(\tau) \bar{w}^T(\alpha)\} e^{A^T(t_k-\alpha)} d\tau d\alpha$$

$\rightarrow \bar{w}(t)$ is white, so $E\{\bar{w}(\tau) \bar{w}^T(\alpha)\} = Q_c(\tau) \delta(\tau-\alpha)$

where $Q_c(t)$ is the time-varying covariance of $\bar{w}(t)$

$\rightarrow$ Apply sifting property of delta function: $\int_a^b x(\alpha) \delta(\tau-\alpha) d\alpha = x(\tau)$

$$\rightarrow P_k = F_{k-1} P_{k-1} F_{k-1}^T + \underbrace{\int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} Q_c(\tau) e^{A^T(t_k-\tau)} d\tau}_{= Q_{k-1}}$$

$$= F_{k-1} P_{k-1} F_{k-1}^T + Q_{k-1}$$

$\rightarrow$ Had to actually calculate Q_{k-1} , but for small Δt , $Q_k \approx Q_c(t_k) \Delta t$